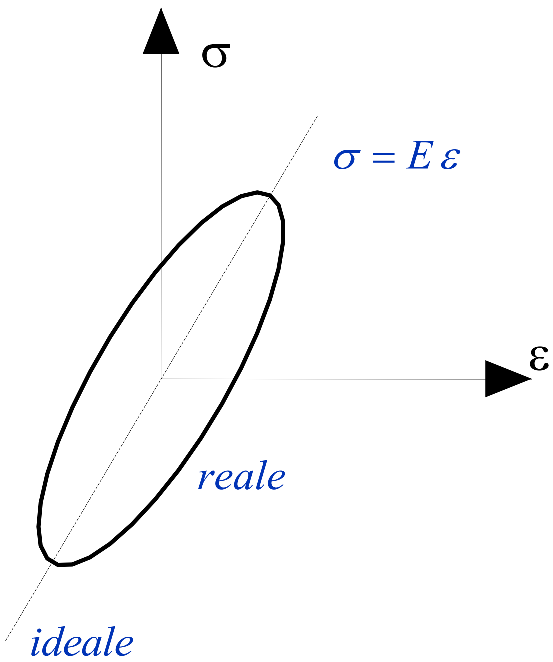
The Modeling of Single-dof Mechanical Systems

Elementary damped oscillator

Elementary damped oscillator

Any real vibrating system dissipates energy because of the phenomena of friction, which is Coulomb, viscous and elastic hysteresis typical of real materials (in nature there are no perfectly elastic bodies which can be applied unconditionally hypothesis Hook).

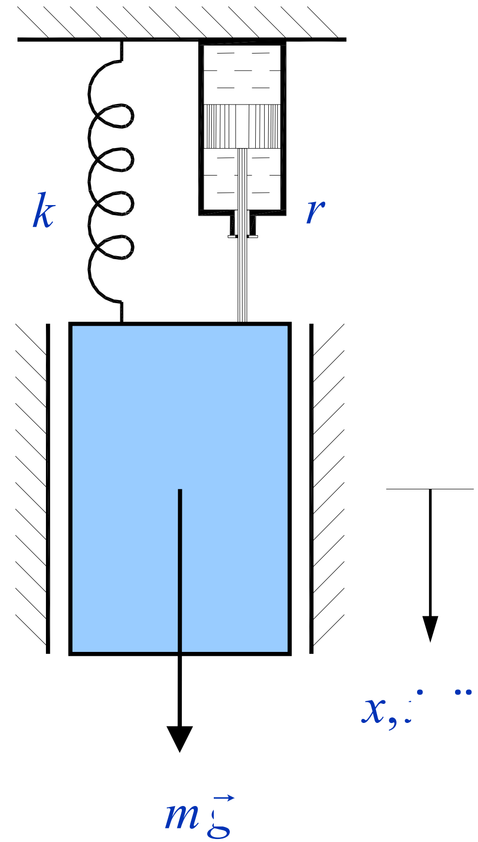
Real bodies subjected to alternating stresses not run through periodically, in the two directions, the straight line that expresses the Hook's law, but run through a curve that inscribes always inside a surface, whose area represents the amount of energy dissipated per cycle because of 'elastic hysteresis.



Elementary damped oscillator

Linear Dashpots

$$F_v = -r\dot{x}$$



Elementary damped oscillator

Dynamic equation:

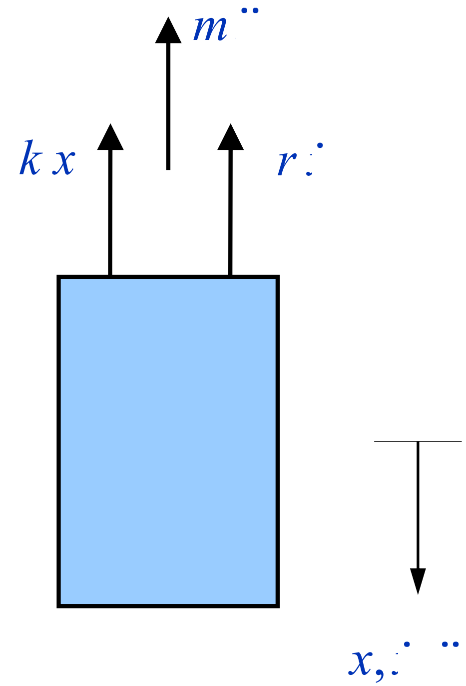
$$-k\,x - r$$

$$m \qquad\qquad 0$$

dividing by m:

$$= 0$$

where: $\alpha = r/2m \qquad \omega = \sqrt{k/m}$



Elementary damped oscillator

The solution of this differential equation of the second order, complete, linear and homogeneous is of the type::

$$x = e^{zt}$$

by substituting in eq. Of motion

$$z^2 + 2\alpha z + \omega^2 = 0$$

whose roots are :

$$z_{1-2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$

according to the value of the discriminant may have three different solutions.

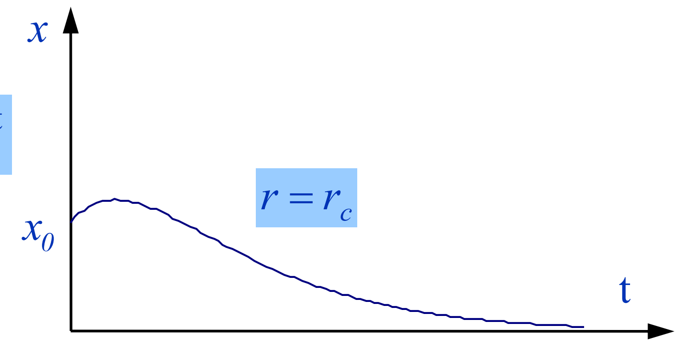
Elementary damped oscillator

1ª solution (critical damping)

$$\alpha^2 - \omega^2 = 0 \quad \Rightarrow \quad \alpha = \omega \quad \Rightarrow \quad \frac{r}{2m} = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad r_c = 2m\omega$$

The parameter r_c, said **critical damping**, the characteristic of the system itself, represents the particular damping value for which it has in the shortest possible time, compared to all other possible values of *r*, the return of the perturbed system in the configuration of static equilibrium.

$$z_1 = z_2 = -\alpha = -\omega \quad \Rightarrow \quad x = Ae^{-\omega t} + Bte^{-\omega t}$$



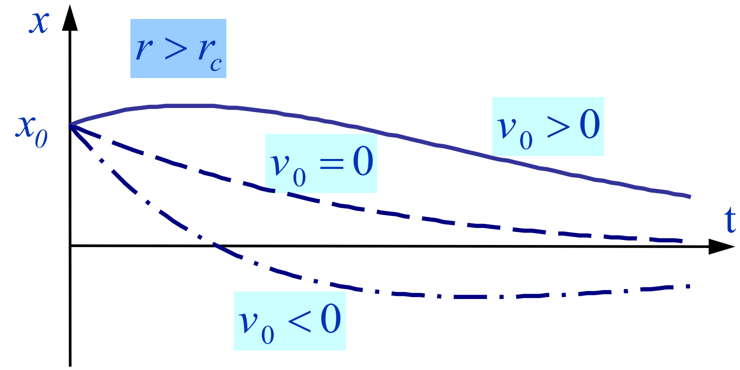
Elementary damped oscillator

2ª solution (Iper-critical systems)

$$\alpha^2 - \omega^2 > 0 \quad \Rightarrow \quad \alpha > \omega \quad \Rightarrow \quad r > r_c$$

The roots of the characteristic equation are real and both negative in the equation of motion becomes:

$$x = Ae^{z_1 t} + Be^{z_2 t}$$



Elementary damped oscillator

3ª solution (Sub-critical systems)

$$\alpha^2 - \omega^2 < 0 \quad \Rightarrow \quad \alpha < \omega \quad \Rightarrow \quad r < r_c$$

This is the case of greatest interest to the mechanical vibrations. The roots of the characteristic equation are complex and conjugate are given by the following equation:

$$z_{1-2} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2} = -\alpha \pm i\beta \qquad \text{con} \qquad \beta = \sqrt{\omega^2 - \alpha^2}$$

eq. becomes:

$$x = Ae^{z_1 t} + Be^{z_2 t}$$

Elementary damped oscillator

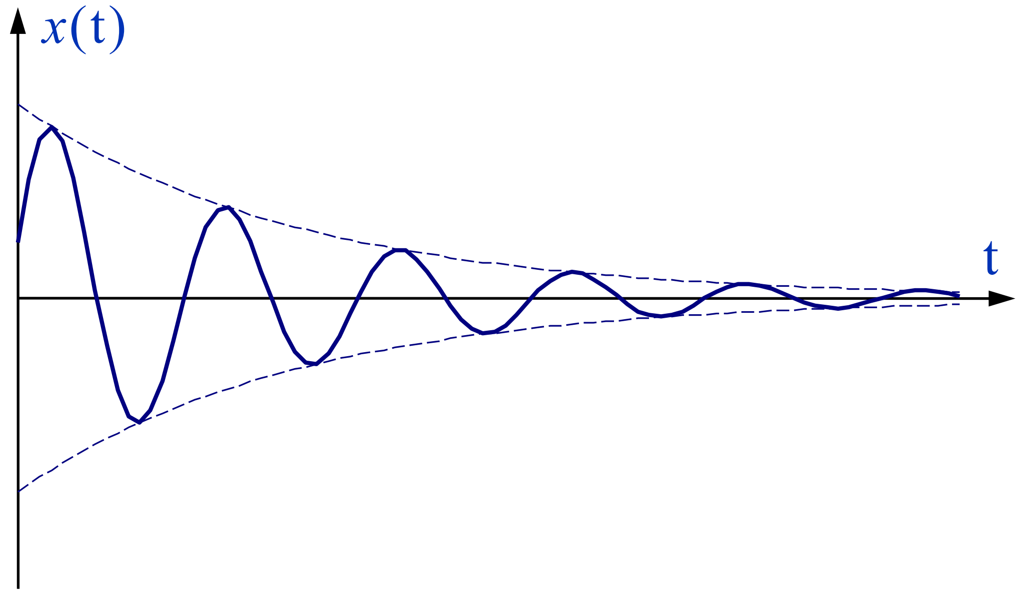
3ª solution (Sub-critical systems)

$$x(\mathrm{t}) = e^{-\alpha \mathrm{t}}\left(A \cos \beta \mathrm{t} + B \, sen \, \beta \mathrm{t}\right)$$

$$x(\mathrm{t}) = e^{-\alpha \mathrm{t}} C \cos(\beta \mathrm{t} + \varphi)$$

$$x(\mathrm{t}) = D e^{(-\alpha + i\beta)\mathrm{t}} + E e^{(-\alpha - i\beta)\mathrm{t}}$$

From the second expression is immediate locate the motion of the mass m is a pseudo motion - periodical, with damped amplitude over time with exponential law.



Elementary damped oscillator

3ª soluzione (Sistemi sub critici)

The natural frequency of the damped system is :

$$\beta = \sqrt{\omega^2 - \alpha^2} = \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2}$$

being generally the term $r/2m$ very small, it can be easily approximated to the natural frequency of the free undamped system:

$$\beta \cong \omega$$

Introducing the dimensionless ratio h and remembering the value of α, the period T of the damped oscillations can be expressed as:

$$h = \frac{r}{r_c} \qquad \alpha = {r}/{2m} \qquad \Rightarrow \qquad T = \frac{2\pi}{\omega\sqrt{1-h^2}}$$

Elementary damped oscillator

Decremento logaritmico

and we can e si basa sulla misura del moto libero smorzato.

If h << 1 and is based on the measurement of free motion damped

We consider: $$x(t) = e^{-\alpha t} C \cos(\beta t + \varphi)$$

$$\frac{x_i}{x_{i+1}}$$ is constant

To have a maximum: $$\cos(\beta t_i + \varphi) = 1$$

then $$\beta t_i + \varphi = 2n\pi$$

We have ∞+1 varing of n: $$t_i = \frac{2n\pi - \varphi}{\beta}$$

Elementary damped oscillator

Decremento logaritmico

the time t_i is the instant which corresponds to a local maximum of $x(t)$, the next maximum will use the instant $(t_i + T)$; then we have:

$$\begin{cases} x(t_i) = Ce^{-\alpha t_i}\cos(\beta t_i + \phi) \\ x(t_i + T) = Ce^{-\alpha(t_i+T)}\cos\left(\beta\left(t_i + T\right) + \phi\right) \end{cases}$$

The ratio of $x(t_i)$ e $x(t_i+T)$:

$$\cos\left(\beta t_i + \phi\right) = \cos\left(\beta\left(t_i + T\right) + \phi\right) \quad \Rightarrow \quad \frac{x(t_i)}{x(t_i + T)} = \frac{Ce^{-\alpha t_i}}{Ce^{-\alpha(t_i+T)}} = e^{\alpha T}$$

then:

$$\delta = \ln\frac{x(t_i)}{x(t_i + T)} = \alpha T$$

logarithmic decrement

Elementary damped oscillator

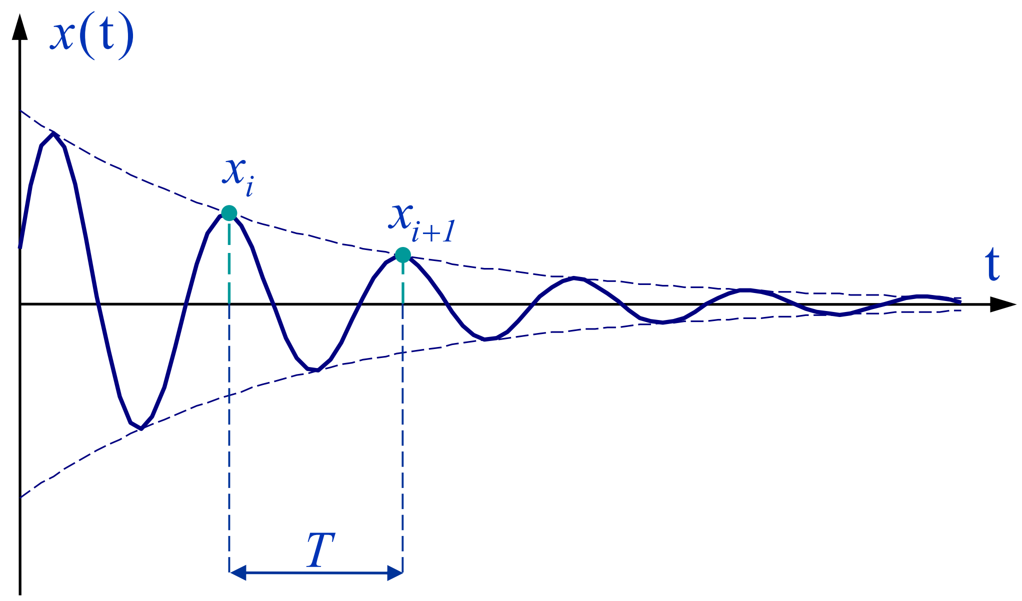
Decremento logaritmico

If it excites any structure with an impulsive forcing is possible to obtain experimentally the diagram in Figure:

it is possible to read the value of T and δ,

And α:

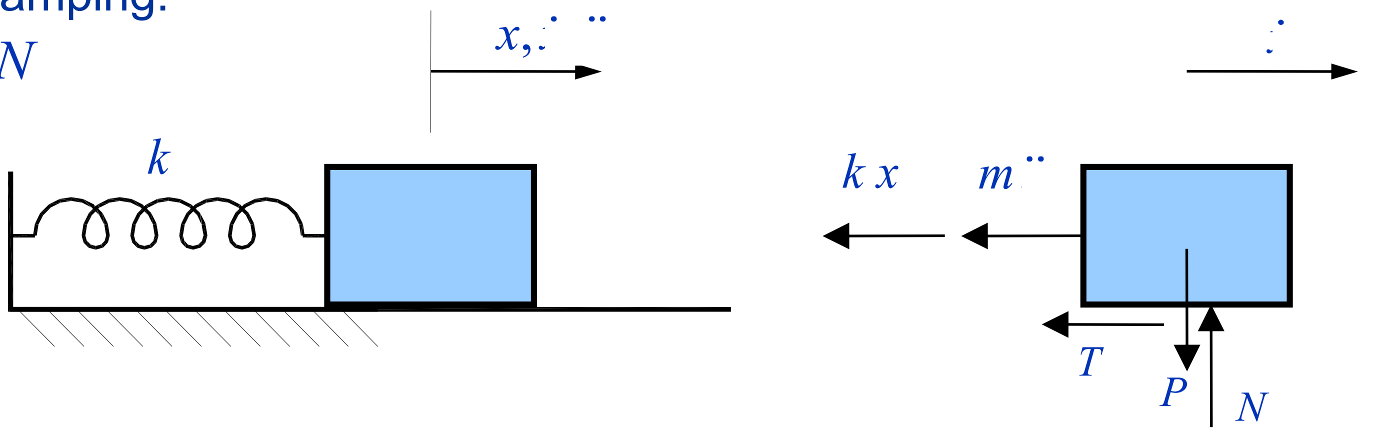
$$\alpha = r/2m$$



Example

Coulomb damping:

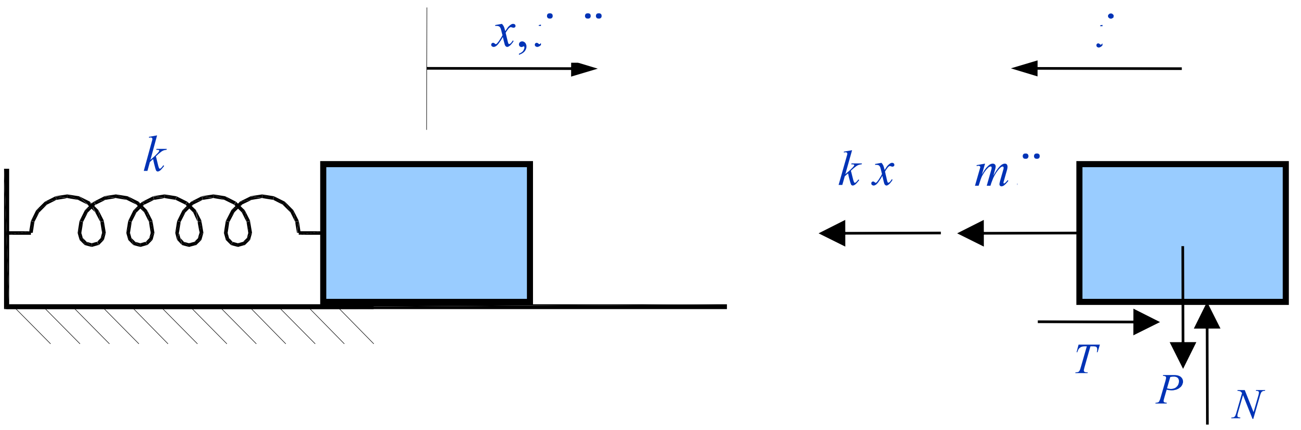
$$T = f_r N$$



$$\begin{cases} x > 0 \quad e \quad x < 0 \\ \dot{\ } \end{cases} \qquad \begin{cases} -kx - T = m\ddot{\ } \\ N = P = mg \end{cases} \qquad m\ddot{\ } \qquad -T \;= -f_r N = -f_r m g$$

$$\ddot{\ } \qquad = -f_r g \quad \text{con} \quad \omega^2 = k/m$$

Example



$$\begin{cases} x > 0 \quad e \quad x < 0 \\ \end{cases}$$

$$\begin{cases} -k\,x + T = m\ddot{} \\ N = P = m\,g \end{cases}$$

$m\ddot{}$

$$T = f_r\,N = f_r\,m\,g$$

$$\ddot{} \qquad = f_r\,g \qquad \text{con} \qquad \omega^2 = k/m$$

Example

$$\begin{cases} x > 0 \quad e \quad x < 0 \\ \dot{\ } \end{cases} \qquad \ddot{\ } \qquad = -f_r\, g \qquad x(\mathrm{t}) = x_g + x_p = C\cos(\omega \mathrm{t} + \varphi) - \frac{T}{k}$$

$$\begin{cases} x > 0 \quad e \quad x < 0 \\ \dot{\ } \end{cases} \qquad \ddot{\ } \qquad = f_r\, g \qquad x(\mathrm{t}) = x_g + x_p = C\cos(\omega \mathrm{t} + \varphi) + \frac{T}{k}$$

where C e φ are constants to be determined by the initial conditions specific to each half-period of the oscillation.

The constant term, denote by Δ, is the arrow static weight:

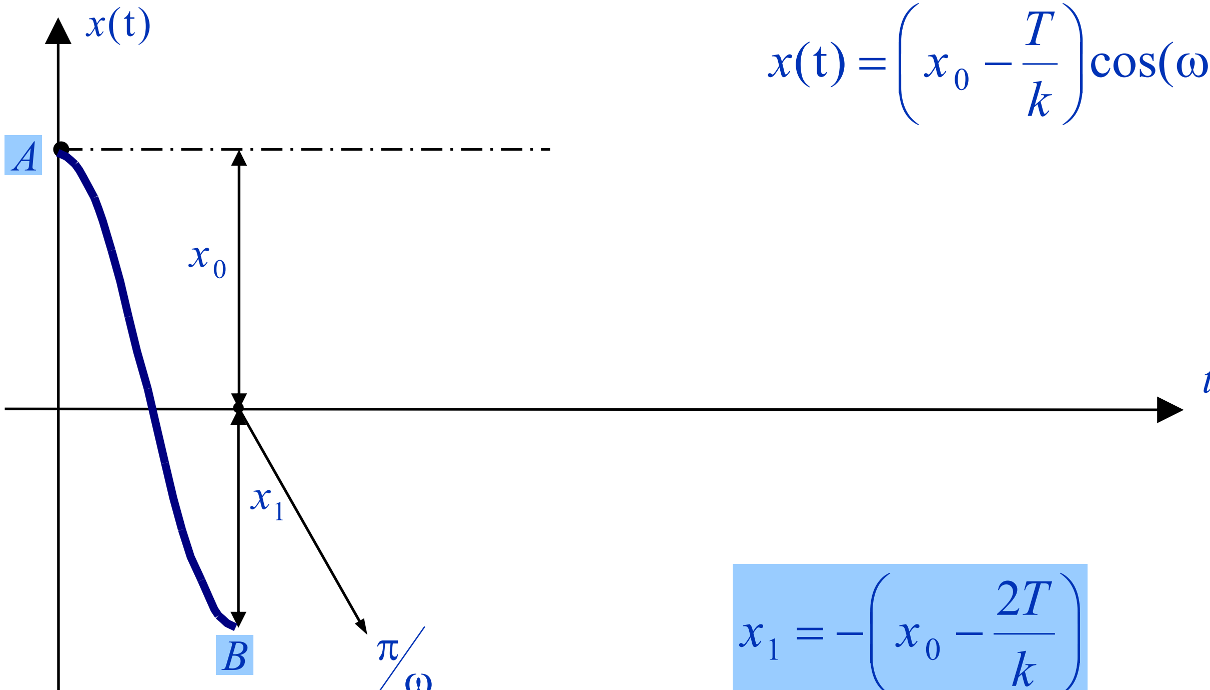
$$\frac{T}{k} = \frac{f_r\, m\, g}{k} = f_r\, \Delta$$

Example

We impose the initial condition:

$$\begin{cases} x > 0 \; e \; x < 0 \\ \vdots \end{cases} \qquad x(t) = C\cos(\omega t + \varphi) + \frac{T}{k} \qquad \begin{cases} x(0) = x_0 \neq 0 \\ \vdots \end{cases} \qquad \begin{cases} C_0 = x_0 - \dfrac{T}{k} \\ \varphi = 0 \end{cases}$$

$$x(t) = \left(x_0 - \frac{T}{k}\right)\cos(\omega t) + \frac{T}{k} \qquad \left(0 \le t \le \pi/\omega\right)$$



$$x\left(t = \frac{\pi}{\omega}\right) = x_1 = $$

$$= \left(x_0 - \frac{T}{k}\right)\cos(\pi) + \frac{T}{k}$$

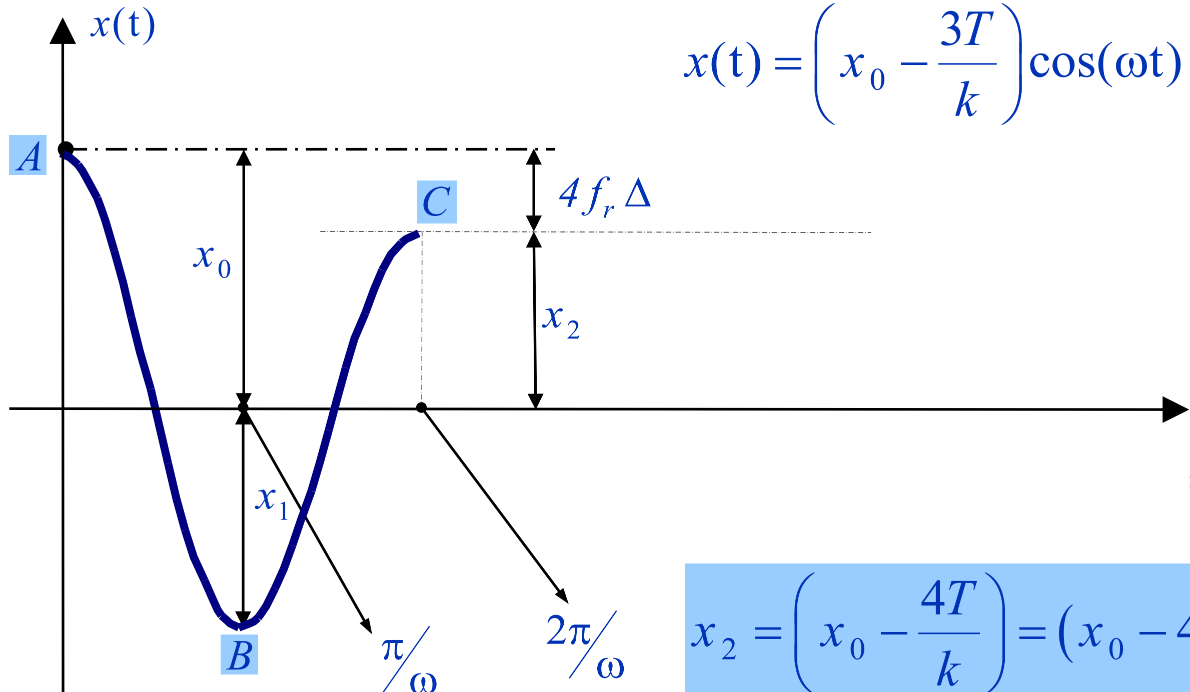
$$x_1 = -\left(x_0 - \frac{2T}{k}\right)$$

Example

We impose the initial condition:

$$\begin{cases} x > 0 \; e \; x < 0 \\ \vdots \end{cases} \qquad x(\mathrm{t}) = C\cos(\omega \mathrm{t} + \varphi) - \frac{T}{k} \qquad \begin{cases} x\left(\frac{\pi}{\omega}\right) = -\left(x_0 - \frac{2T}{k}\right) \\ \vdots \end{cases} \qquad \begin{cases} C_1 = x_0 - \frac{3T}{k} \\ \varphi = 0 \end{cases}$$

$$x(\mathrm{t}) = \left(x_0 - \frac{3T}{k}\right)\cos(\omega \mathrm{t}) - \frac{T}{k} \qquad \left(\pi/\omega \le \mathrm{t} \le 2\pi/\omega\right)$$



$$x\left(\mathrm{t} = \frac{2\pi}{\omega}\right) = x_2 =$$

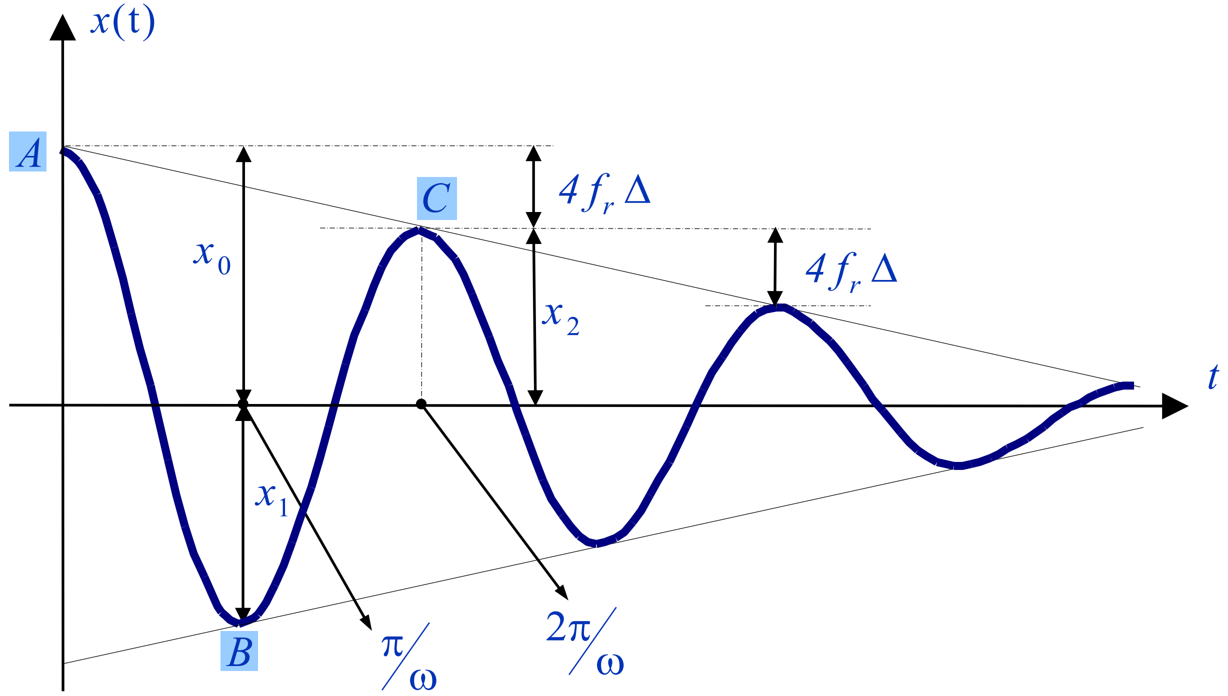
$$= \left(x_0 - \frac{3T}{k}\right)\cos(2\pi) - \frac{T}{k}$$

$$x_2 = \left(x_0 - \frac{4T}{k}\right) = \left(x_0 - 4 f_r \Delta\right)$$

Example

Therefore, the reduction in amplitude is $\frac{4T}{k} = 4 f_r \Delta$ for each oscillation complet.

The found value of x_2 becomes the initial condition for the third half-period, and the process can be repeated until the end of the motion..

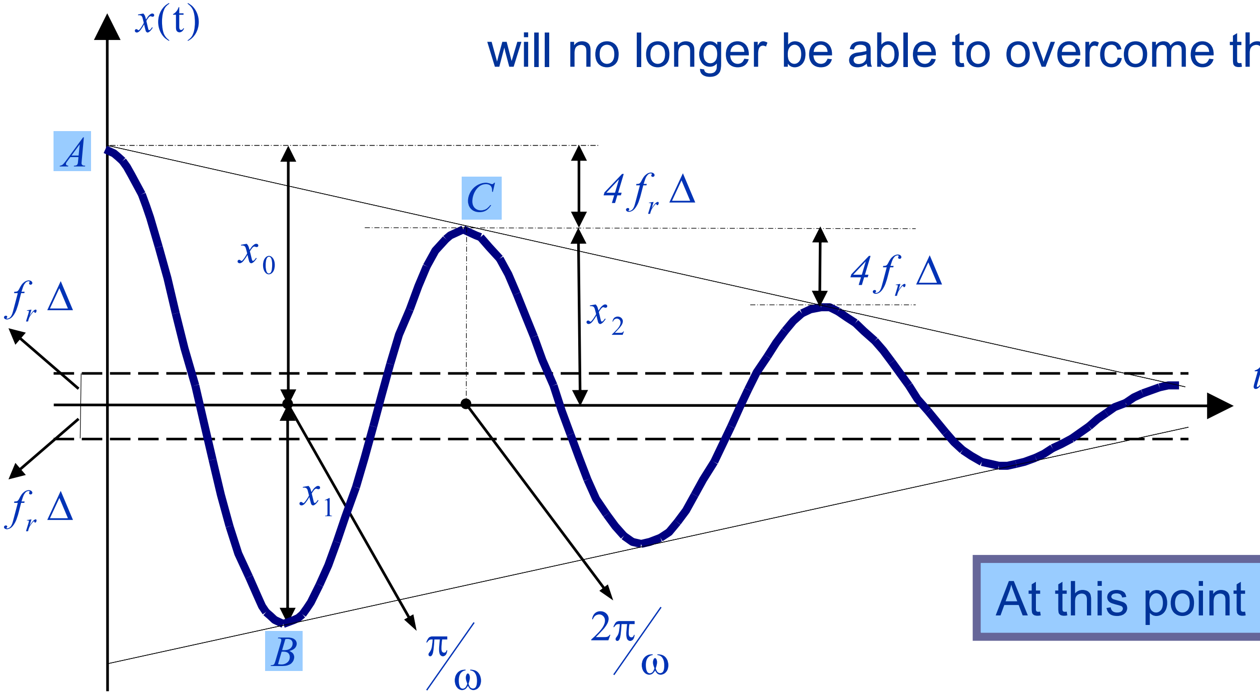


Example

When the amplitude of $x(\mathrm{t})$ will be reduced to a value less than:

$$\frac{T}{k} = f_r\,\Delta \qquad \text{cioè} \qquad x(\mathrm{t}) < x_f = f_r\,\Delta$$

then the force of elastic of the spring, amounting to kx_f will no longer be able to overcome the force of friction T



At this point the motion stops..