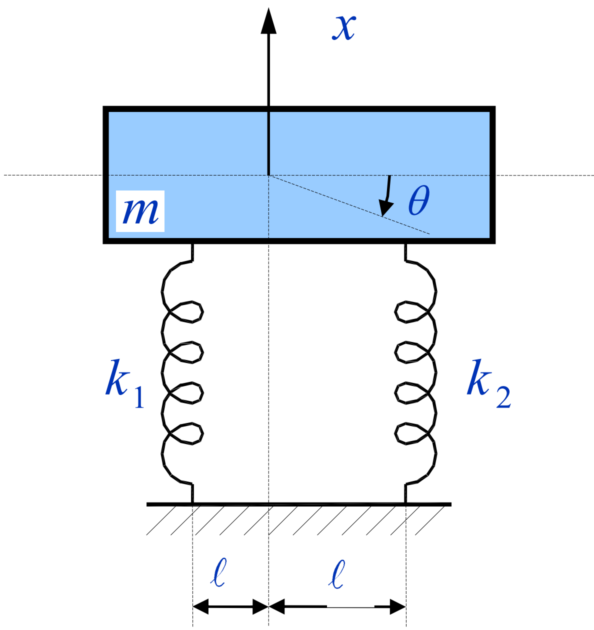
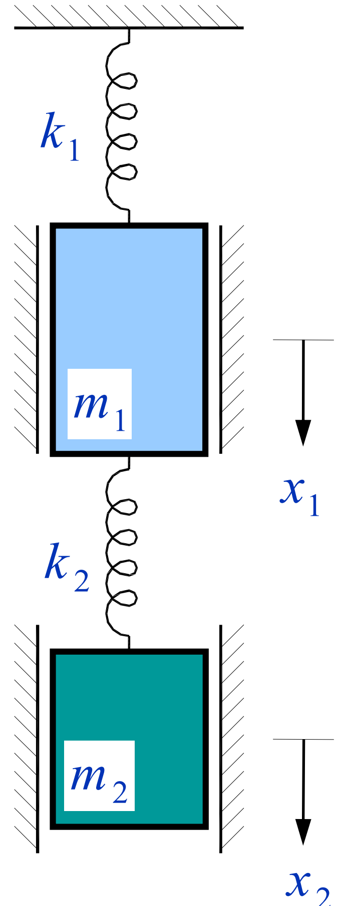
The Modeling of Two-dof Mechanical Systems

Free vibrations

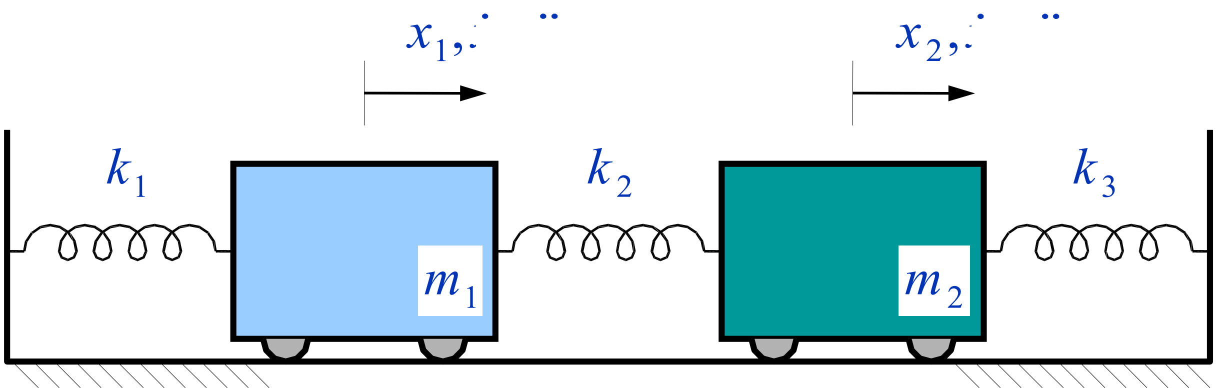
Examples of modeling of two-dof of mechanical systems.





The Modeling of Two-dof Mechanical Systems

Referring to Figure choose the two independent linear coordinates x1 and x2 from the static equilibrium position of the two masses m1 and m2.



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We write the equations of dynamic equilibrium for the two masses::

Mass 1

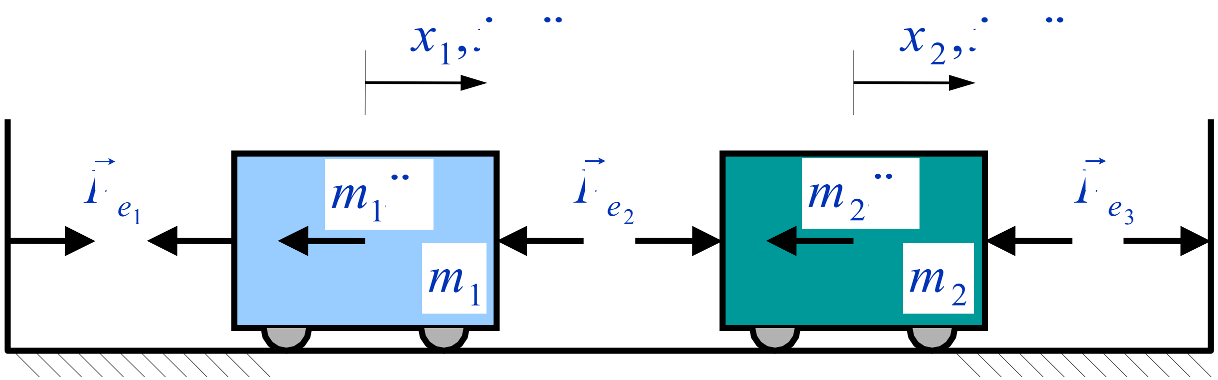
$$\vec{F}_{e_1}+\vec{F}_{e_2}=m_1\ddot{x}_1$$

$$-k_1x_1-k_2\left(x_1-x_2\right)=m_1\ddot{x}_1$$

Mass 2

$$\vec{F}_{e_2}+\vec{F}_{e_3}=m_2\ddot{x}_2$$

$$-k_2\left(x_2-x_1\right)-k_3x_2=m_2\ddot{x}_2$$



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The system of equation of motion becomes:

$$\begin{cases} -m_1 \ddot{\ } \qquad -k_2\left(x_1 - x_2\right) = 0 \\ -m_2 \ddot{\ } \qquad x_2 - x_1\right) - k_3 x_2 = 0 \end{cases}$$

and:

$$\begin{cases} m_1 \ddot{\ } \qquad + k_2\right) x_1 - k_2 x_2 = 0 \\ m_2 \ddot{\ } \qquad {}_1 + \left(k_2 + k_3\right) x_2 = 0 \end{cases}$$

i.e. a system of homogeneous linear differential equations of second order with constant coefficients, coupled to the term k_2.

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In matrix notation:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{} \\ \ddot{} \end{Bmatrix} \qquad \begin{bmatrix} {}_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

And in compact form:

$$\underset{[2x2]}{\mathbf{M}}\ \underset{[2x1]}{\ddot{}} \qquad \underset{[2x2]}{} \underset{[2x1]}{\mathbf{x}} = \underset{[2x1]}{\mathbf{0}}$$

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The system accepts solutions of the type:

$$\begin{cases} x_1 = X_1 e^{z\,t} \\ x_2 = X_2 e^{z\,t} \end{cases} \qquad \begin{cases} \dot{\ } \quad \quad e^{z\,t} \\ \dot{\ } \quad \quad e^{z\,t} \end{cases} \qquad \begin{cases} \ddot{\ } \quad \quad X_1 e^{z\,t} \\ \ddot{\ } \quad \quad X_2 e^{z\,t} \end{cases}$$

Substituting in the system has:

$$\begin{cases} X_1\left(m_1 z^2 + k_1 + k_2\right) + X_2\left(-k_2\right) = 0 \\ X_1\left(-k_2\right) + X_2\left(m_2 z^2 + k_2 + k_3\right) = 0 \end{cases}$$

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The algebraic system admits solutions different from the banal if and only if the determinant of the matrix of coefficients is equal to zero, ie:

$$\Delta = \begin{vmatrix} m_1 z^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 z^2 + k_2 + k_3 \end{vmatrix} = 0$$

obtaining the following equation in biquadratic z:

$$\Delta = m_1 m_2 z^4 + \left[m_1 (k_2 + k_3) + m_2 (k_1 + k_2) \right] z^2 + (k_1 + k_2)(k_2 + k_3) - k_2^2$$

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in general:

$$a_1 z^4 + a_2 z^2 + a_3 = 0$$

whose solutions are valid:

$$z_{1-2}^2 = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$$

It can be shown that the radicand is always positive and that its root is always less than a2, then it follows that the roots are both z_{1-2}^2 negative, and then you have four imaginary roots, two by two conjugated.

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At the two solutions found, the system is:

$$\begin{cases} X_1\left(m_1 z^2 + k_1 + k_2\right) + X_2\left(-k_2\right) = 0 \\ X_1\left(-k_2\right) + X_2\left(m_2 z^2 + k_2 + k_3\right) = 0 \end{cases}$$

having determinant zero, is reduced to a single equation of the two present, being a linear combination of the other.

It is not possible to determine the constants X_1 e X_2, but only and only their relationship.

for $z = z_1 \quad \Rightarrow \quad \dfrac{X_1(z_1)}{X_2(z_1)} = \dfrac{k_2}{m_1 z_1^2 + k_1 + k_2} = \mu_1(z_1) \quad \Rightarrow \quad X_1(z_1) = \mu_1 X_2(z_1)$

for $z = z_2 \quad \Rightarrow \quad \dfrac{X_1(z_2)}{X_2(z_2)} = \dfrac{k_2}{m_1 z_2^2 + k_1 + k_2} = \mu_2(z_2) \Rightarrow \quad X_1(z_2) = \mu_2 X_2(z_2)$

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Therefore, the general solution of the system of differential equations is a linear combination of the two solutions corresponding to $z = z_1$ ed a $z = z_2$, so we can, wirte:

$$\begin{cases} x_1(\mathrm{t}) = X_1(z_1)e^{z_1\mathrm{t}} + X_1(z_2)e^{z_2\mathrm{t}} \\ x_2(\mathrm{t}) = X_2(z_1)e^{z_1\mathrm{t}} + X_2(z_2)e^{z_2\mathrm{t}} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1(\mathrm{t}) = \mu_1 X_2(z_1)e^{z_1\mathrm{t}} + \mu_2 X_2(z_2)e^{z_2\mathrm{t}} \\ x_2(\mathrm{t}) = X_2(z_1)e^{z_1\mathrm{t}} + X_2(z_2)e^{z_2\mathrm{t}} \end{cases}$$

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being z_{1-2}^{2} the imaginary can be put :

$$z_1 = \pm i\omega_1 \quad z_2 = \pm i\omega_2$$

by the Euler equations, we can write the integral in the following form:

$$\begin{cases} x_1(\mathrm{t}) = \mu_1 A \,\mathrm{sen}(\omega_1 \mathrm{t} + \psi_1) + \mu_2 B \,\mathrm{sen}(\omega_2 \mathrm{t} + \psi_2) \\ x_2(\mathrm{t}) = A \,\mathrm{sen}(\omega_1 \mathrm{t} + \psi_1) + B \,\mathrm{sen}(\omega_2 \mathrm{t} + \psi_2) \end{cases}$$

To determine the four arbitrary constants A, B, ψ_1 e ψ_2 must impose four initial conditions, for t = 0:

$$\begin{cases} x_1(0) = x_{1_0} \\ \\ x_2(0) = x_{2_0} \\ \end{cases}$$

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By imposing that the initial conditions satisfy the following relationship:

$$\frac{x_{1_0}}{x_{2_0}} = \frac{v_{1_0}}{v_{2_0}} = \mu_1 \quad \Rightarrow \quad B = 0$$

the general integral becomes:

$$\begin{cases} x_1(t) = \mu_1 A \operatorname{sen}(\omega_1 t + \psi_1) \\ x_2(t) = A \operatorname{sen}(\omega_1 t + \psi_1) \end{cases}$$

Then the system with two degrees of freedom, begins to vibrate sinusoidally with pulsation ω_1 as if he had a single degree of freedom, and this vibration, pulsation with ω_1 and the amplitude ratio μ_1 constant, is called the ***first mode of vibration of the system.***

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By imposing that the initial conditions satisfy the following relationship:

$$\frac{x_{1_0}}{x_{2_0}} = \frac{v_{1_0}}{v_{2_0}} = \mu_2 \quad \Rightarrow \quad A = 0$$

the general integral becomes:

$$\begin{cases} x_1(\mathrm{t}) = \mu_2 B \,\mathrm{sen}(\omega_2 \mathrm{t} + \psi_2) \\ x_2(\mathrm{t}) = B \,\mathrm{sen}(\omega_2 \mathrm{t} + \psi_2) \end{cases}$$

Then the system with two degrees of freedom, begins to vibrate sinusoidally with pulsation ω_2 as if he had a single degree of freedom, and this vibration, pulsation with ω_2 and the amplitude ratio constant μ_2, is defined according to its own way of vibrating system ***second mode of vibration of the system***.

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If the initial conditions are generic system will vibrate according to the sum of two sinusoids, as shown in figure: