The Modeling of Two-dof Mechanical Systems

forced oscillations

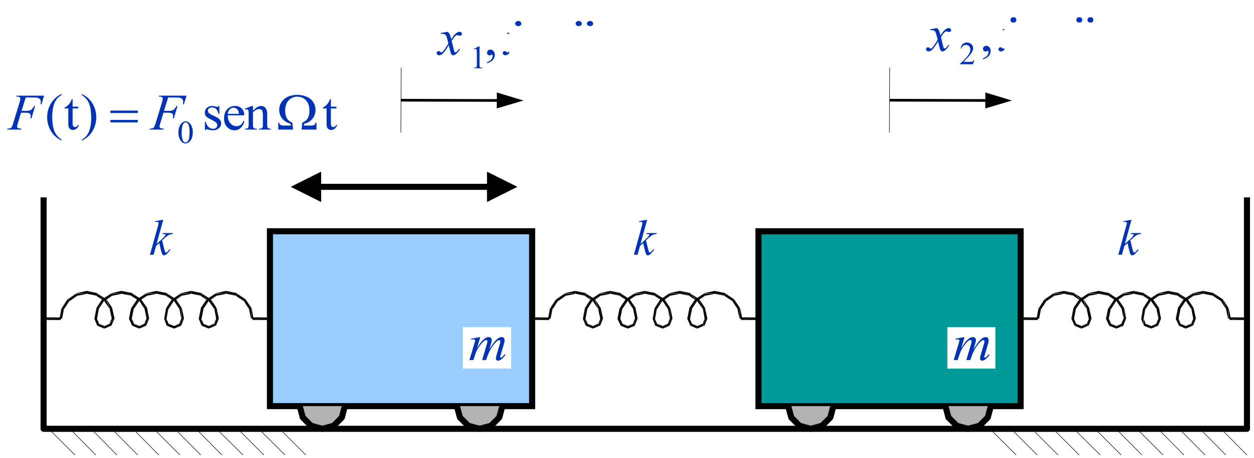
Sistema a due gradi di libertà forzato

Let's analyze the following scheme in the particular case of:

$$m_1 = m_2 = m \qquad k_1 = k_2 = k_3 = k$$

With the force F(t):

$$F(t) = F_0 \, \mathrm{sen}\, \Omega t$$



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The system of dynamic equilibrium equation becomes:

$$\begin{cases} m_1 \ddot{} \quad + k_2) x_1 - k_2 x_2 = F_0 \,\mathrm{sen}\,\Omega \mathrm{t} \\ m_2 \ddot{} \quad {}_1 + (k_2 + k_3) x_2 = 0 \end{cases} \qquad \begin{cases} m \ddot{} \quad x_1 - k x_2 = F_0 \,\mathrm{sen}\,\Omega \mathrm{t} \\ m \ddot{} \quad + 2k x_2 = 0 \end{cases}$$

In matrix notation:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{} \\ \ddot{} \end{Bmatrix} \quad \begin{bmatrix} 2k & -k_2 \\ -k_2 & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \,\mathrm{sen}\,\Omega \mathrm{t} \\ 0 \end{Bmatrix}$$

And in compact form:

$$\underset{[2x2][2x1]}{\mathbf{M}\ddot{}} \quad \underset{[2x2][2x1]}{\mathbf{x}} = \underset{[2x1]}{\mathbf{F}}$$

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The solutions

$$\begin{cases} x_1 = x_{1gen} + x_{1part} \\ x_2 = x_{2gen} + x_{2part} \end{cases}$$

Let us first study the general integral of homogeneous associated with give us the free vibrations of the system (in the absence of forcing), it is of the type:

$$\begin{cases} x_{1gen} = X_1 \,\mathrm{sen}\left(\omega t + \psi\right) \\ x_{2gen} = X_2 \,\mathrm{sen}\left(\omega t + \psi\right) \end{cases}$$

and deriving and substituting in the homogeneous system associated, is obtained:

$$\begin{cases} X_1\left(m\omega^2 - 2k\right) + X_2 k = 0 \\ X_1 k + X_2\left(m\omega^2 - 2k\right) = 0 \end{cases}$$

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The algebraic system admits solutions different from the trivial if and only if the determinant of the matrix of coefficients is equal to zero, ie:

$$\Delta = \begin{vmatrix} m\omega^2 - 2k & k \\ k & m\omega^2 - 2k \end{vmatrix} = 0$$

resulting in the following equation biquadratic ω:

$$\Delta = m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

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And the natual frequency:

$$\omega_{1-2}^{2}=\frac{2km\pm\sqrt{4k^{2}m^{2}-3k^{2}m^{2}}}{m^{2}}\quad\Rightarrow\quad\begin{cases}\omega_{1}=\sqrt{\dfrac{k}{m}}\\ \omega_{2}=\sqrt{\dfrac{3k}{m}}\end{cases}$$

per $\omega=\omega_{1}\quad\Rightarrow\quad\dfrac{X_{1}(\omega_{1})}{X_{2}(\omega_{1})}=\dfrac{-k}{m\omega_{1}^{2}-2k}=\mu_{1}=1$

have the first modo of vibration with pulsation ω_1 and law of motion:

$$\begin{cases}x_{1gen}=\mu_{1}X_{2}(\omega_{1})\mathrm{sen}(\omega_{1}\mathrm{t}+\psi_{1})\\ x_{2gen}=X_{2}(\omega_{1})\mathrm{sen}(\omega_{1}\mathrm{t}+\psi_{1})\end{cases}$$

being amplitudes equal $X_1 = X_2$, the two carts are moving as if they were rigidly connected, the central spring is not deformed during the first mode of vibration,

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per $\omega = \omega_2 \quad \Rightarrow \quad \frac{X_1(\omega_2)}{X_2(\omega_2)} = \frac{-k}{m\omega_2^2 + 2k} = \mu_2 = -1$

We have the II mode of vibrations with ω_2 e low of motion:

$$\begin{cases} x_{1gen} = \mu_2 X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) \\ x_{2gen} = X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) \end{cases}$$

amplitudes of motion of the two carts still have the same form but opposite, the side springs are deformed in equal measure while the center has a double .

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If there were no external forcing the generic free motion of the system would be expressed by:

$$\begin{cases} x_{1gen} = X_2(\omega_1)\mathrm{sen}(\omega_1 t + \psi_1) - X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) \\ x_{2gen} = X_2(\omega_1)\mathrm{sen}(\omega_1 t + \psi_1) + X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) \end{cases}$$

The four constants: $\left[X_2(\omega_1) \quad X_2(\omega_2) \quad \psi_1 \quad \psi_2 \right]$

For t = 0 :

$$\begin{cases} x_1(0) = x_{1_0} = X_2(\omega_1)\mathrm{sen}\psi_1 - X_2(\omega_2)\mathrm{sen}\psi_2 \\ \quad = \omega_1 X_2(\omega_1)\cos\psi_1 - \omega_2 X_2(\omega_2)\cos\psi_2 \\ x_2(0) = x_{2_0} = X_2(\omega_1)\mathrm{sen}\psi_1 + X_2(\omega_2)\mathrm{sen}\psi_2 \\ \quad = \omega_1 X_2(\omega_1)\cos\psi_1 + \omega_2 X_2(\omega_2)\cos\psi_2 \end{cases}$$

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Let us now study the particular integral of the system from which forced the motion to the system, that is, when the vibration is damped transient initial effect of the causes of dissipative always present in real systems.

The particular integral is of the type:

$$\begin{cases} x_{1part} = A \operatorname{sen}(\Omega t + \varphi) \\ x_{2part} = B \operatorname{sen}(\Omega t + \varphi) \end{cases}$$

that derivative and substituted into the system:

$$\begin{cases} m\ddot{} \quad \ldots \quad x_1 - k x_2 = F_0 \operatorname{sen}\Omega t \\ m\ddot{} \quad \ldots \quad + 2k x_2 = 0 \end{cases} \Rightarrow \begin{cases} A\left(m\Omega^2 - 2k\right) + Bk = -F_0 \\ Ak + B\left(m\Omega^2 - 2k\right) = 0 \end{cases}$$

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with Cramer's rule, we obtain the values of the amplitudes:

$$A = \frac{\Delta_1}{\Delta} \qquad\qquad B = \frac{\Delta_2}{\Delta}$$

with:

$$\Delta_1 = \begin{vmatrix} -F_0 & k \\ 0 & m\Omega^2 - 2k \end{vmatrix} = F_0\left(2k - m\Omega^2\right)$$

$$\Delta_2 = \begin{vmatrix} m\Omega^2 - 2k & -F_0 \\ k & 0 \end{vmatrix} = F_0 k$$

The values of the amplitudes A and B of the integrals of motion particular, amplitudes that scheme will be merged with those of the absolute motion, can be represented as a function of pulsation Ω of external forcing.

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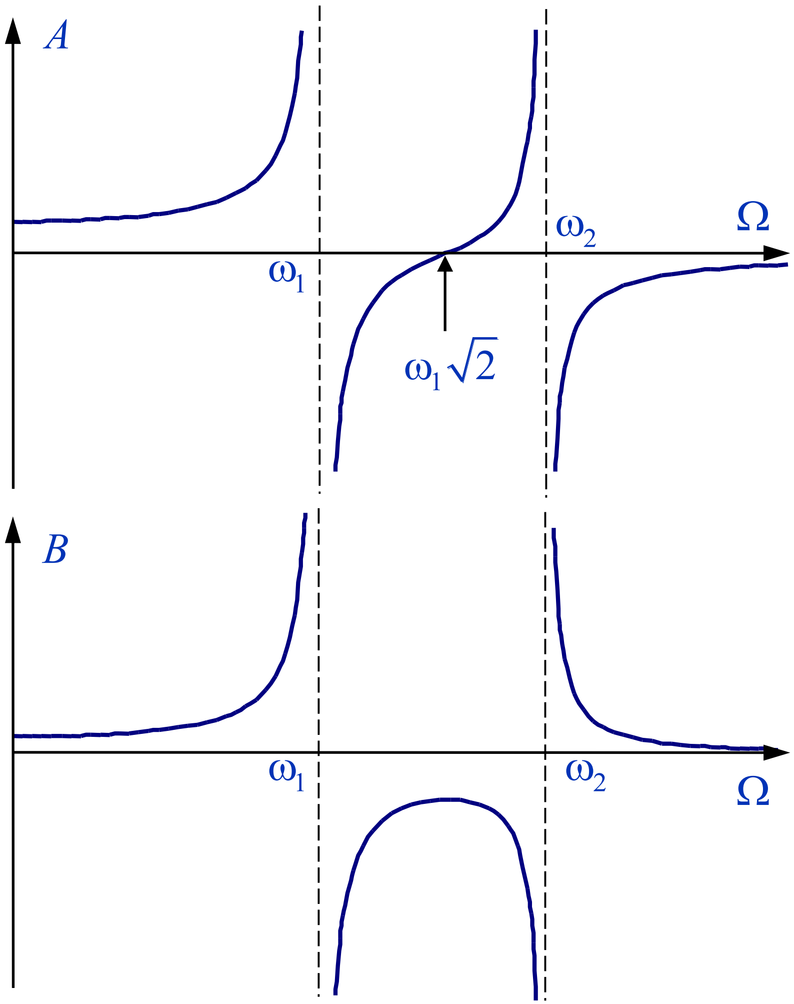
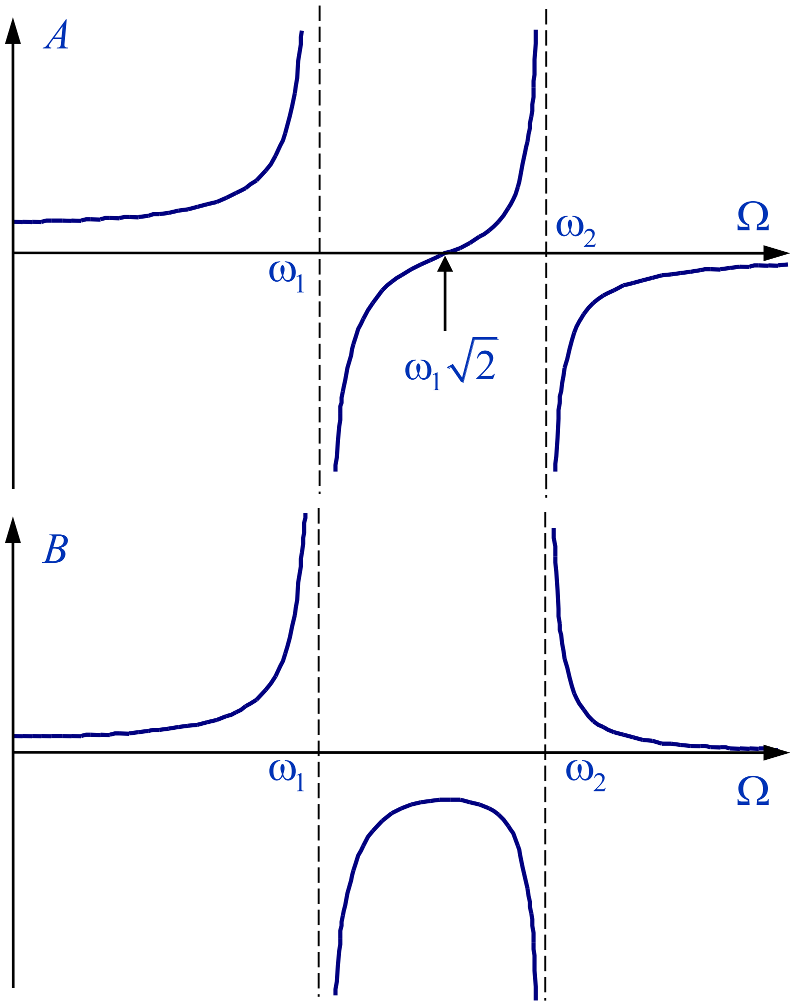


$\Omega << \omega_1$

the motion of the two masses is concordant and in phase with the forcing; amplitudes are almost equal to the corresponding static displacements; deformations of the springs in practice are those that would have been obtained by applying to the mass a constant force F_0.

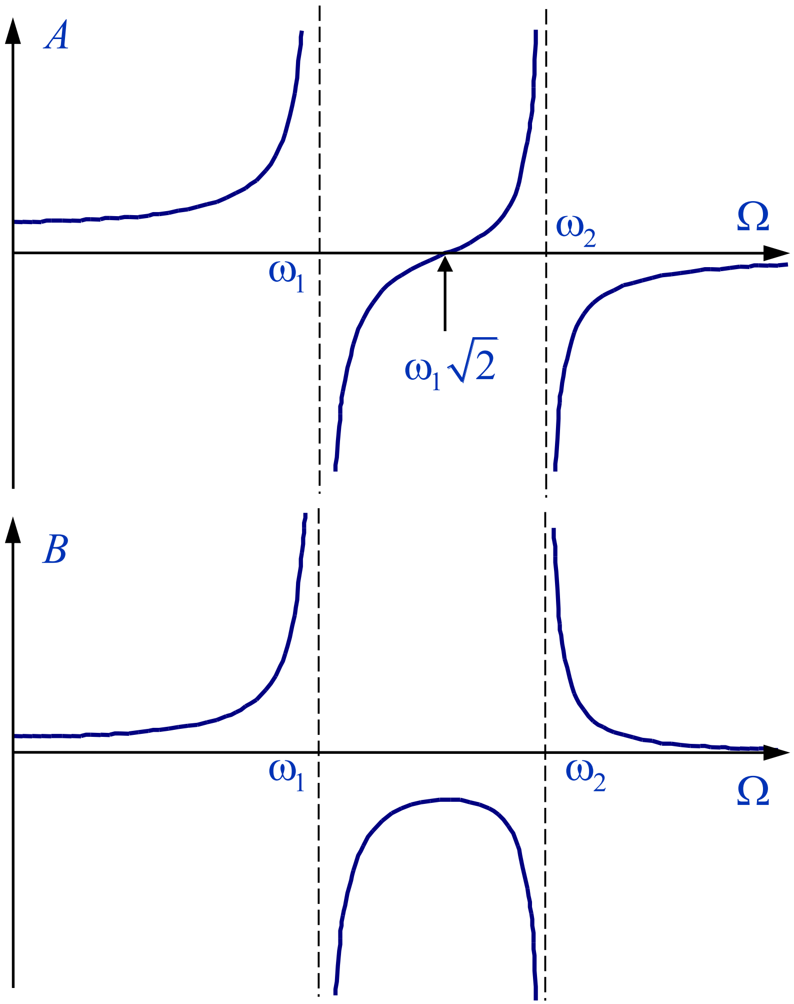
Sistema a due gradi di libertà forzato



$\Omega \to \omega_1$

il moto dei due carrellini è ancora equiverso con ampiezze di notevole entità che tendono a diventare praticamente uguali tra di loro e, per $\Omega < \omega_1$, il moto è ancora in fase con la forzante, mentre è in controfase per $\Omega > \omega_1$; la molla centrale si deforma molto meno di quelle laterali: la deformata di vibrazione del sistema è praticamente uguale a quella del I modo proprio di vibrare, siamo in presenza della prima pulsazione di risonanza del sistema.

Sistema a due gradi di libertà forzato



$\omega_1 < \Omega < \sqrt{2}\,\omega_1$

gli spostamenti sono equiversi tra di loro, ma in opposizione di fase rispetto alla forzante; le ampiezze diminuiscono anche se x_{2part} rimane, in valore assoluto, maggiore di x_{1part}.

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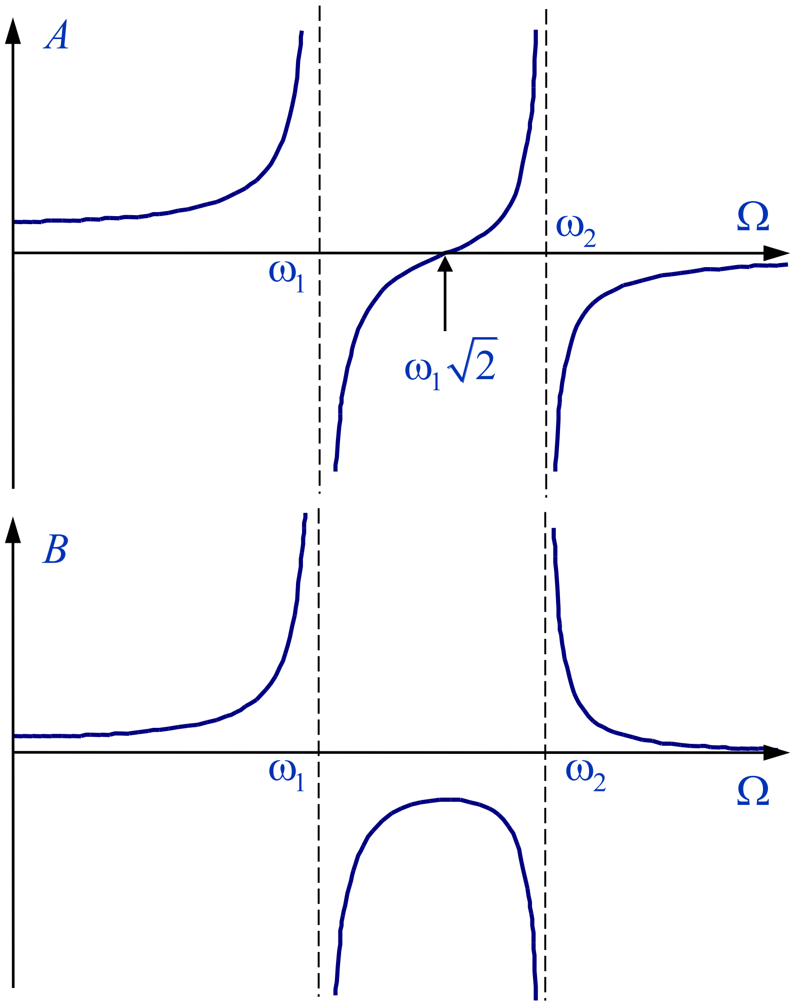

$\Omega = \sqrt{2}\,\omega_1$

la massa a cui è applicata la forzante resta teoricamente ferma; l'altra massa si muove quel tanto che basta per far si che la molla centrale trasmetta alla prima una forza esattamente uguale e contraria alla forzante esterna.

Questo particolare fenomeno viene sfruttato per smorzare le vibrazioni delle macchine.

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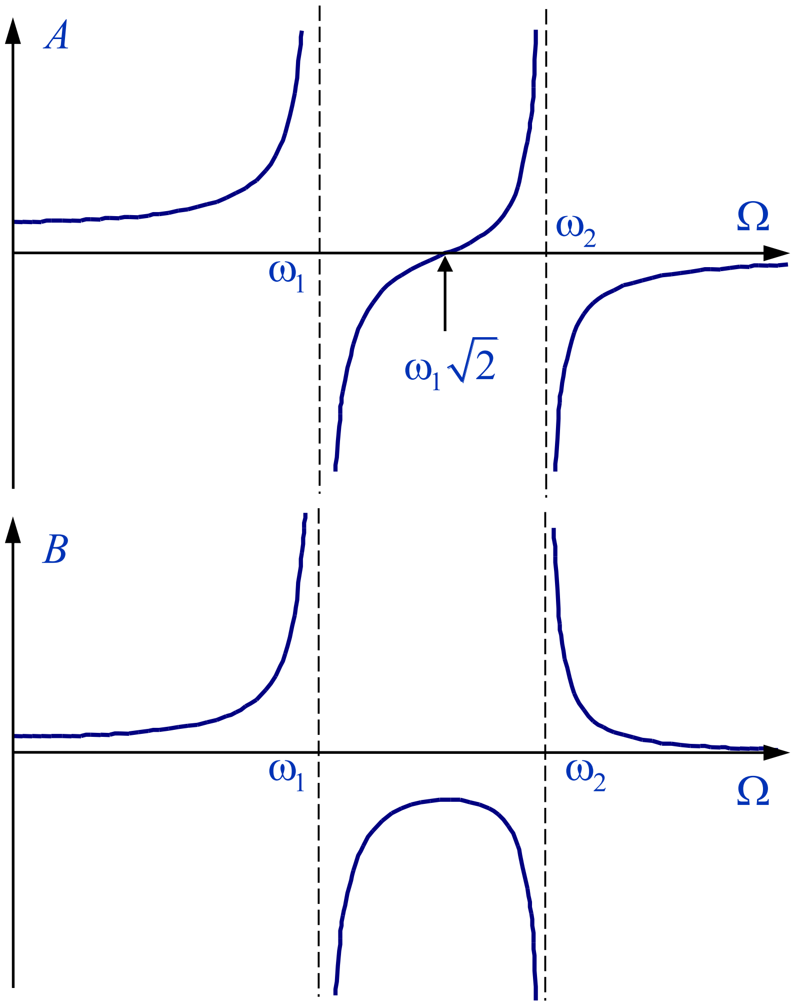
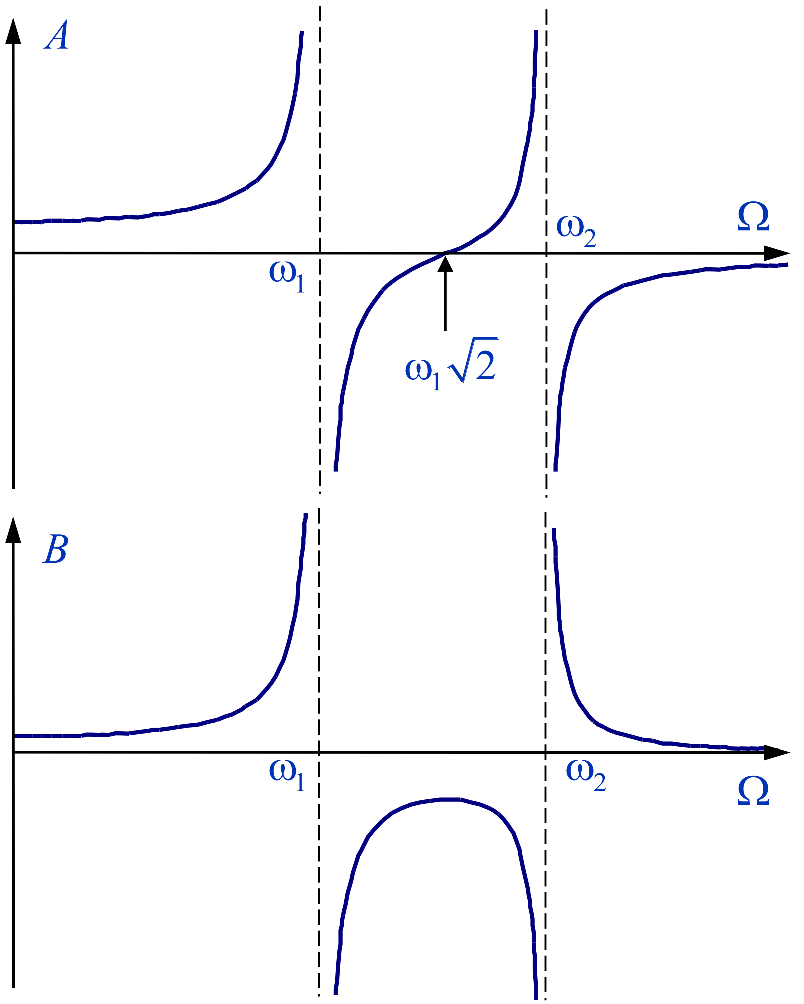


$\sqrt{2}\,\omega_1 < \Omega < \omega_2$

gli spostamenti hanno segno opposto, il primo in fase con la forzante, il secondo in opposizione; le rispettive ampiezze, dapprima piccole, aumentano all'aumentare di Ω.

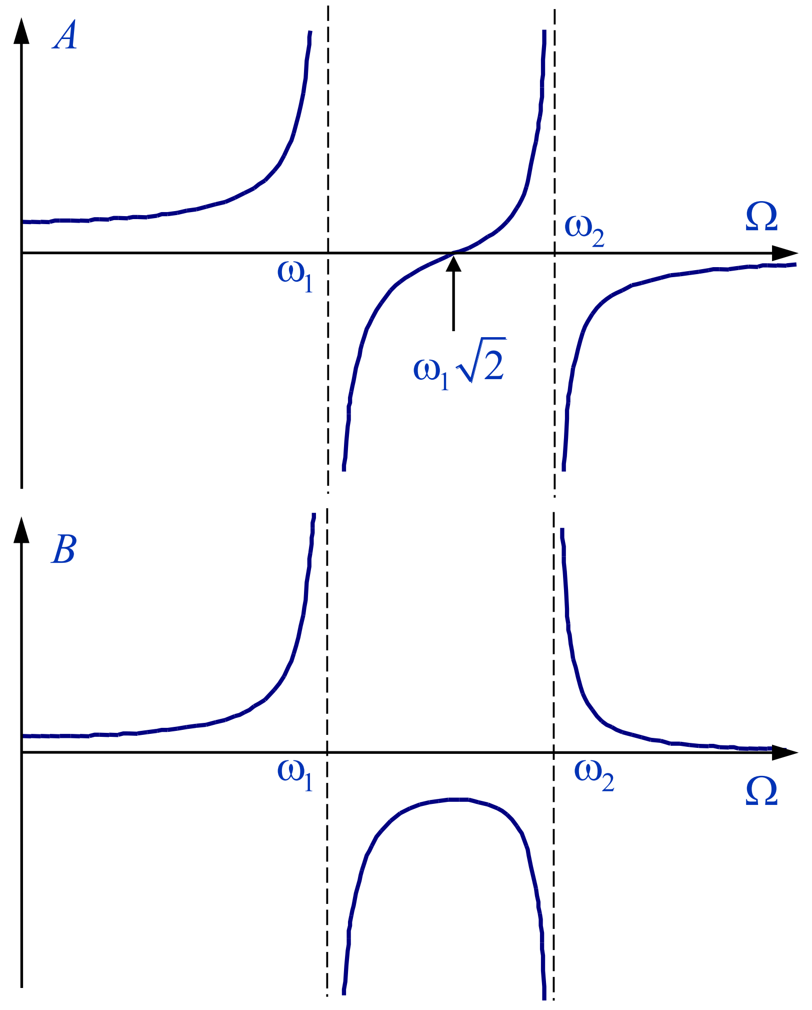
Sistema a due gradi di libertà forzato



$\Omega \rightarrow \omega_2$

gli spostamenti x_{1part} e x_{2part} tornati ad essere molto grandi e di egual valore, hanno segno opposto; la molla centrale si deforma circa il doppio di quelle laterali, ed un suo punto resta fermo; la deformata di vibrazione del sistema è praticamente uguale a quella del II modo proprio di vibrare, siamo alla seconda risonanza.

Sistema a due gradi di libertà forzato



$\Omega >> \omega_2$

entrambi gli spostamenti, pur avendo segno opposto, tendono rapidamente a zero all'aumentare di Ω; il sistema in pratica non si deforma più e rimane in quiete.

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The law of motion of the system forced, in its initial transient, is given by the complete:

$$\begin{cases} x_1 = X_2(\omega_1)\mathrm{sen}(\omega_1 t + \psi_1) - X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) + A\,\mathrm{sen}(\Omega t + \varphi) \\ x_2 = X_2(\omega_1)\mathrm{sen}(\omega_1 t + \psi_1) + X_2(\omega_2)\mathrm{sen}(\omega_2 t + \psi_2) + B\,\mathrm{sen}(\Omega t + \varphi) \end{cases}$$

The values: $\begin{bmatrix} X_2(\omega_1) & X_2(\omega_2) & \psi_1 & \psi_2 \end{bmatrix}$

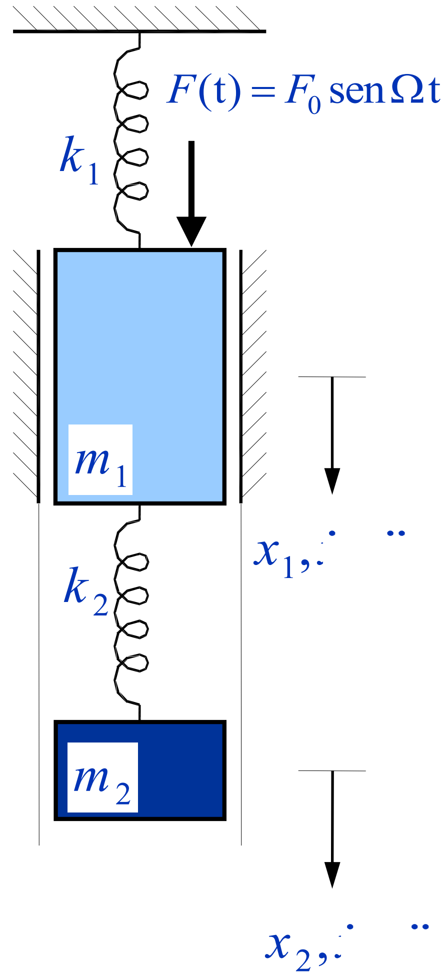
the initial conditions:

$$\begin{cases} x_1(0) = x_{1_0} = X_2(\omega_1)\mathrm{sen}\psi_1 - X_2(\omega_2)\mathrm{sen}\psi_2 + A\,\mathrm{sen}\,\varphi \\ \qquad = \omega_1 X_2(\omega_1)\cos\psi_1 - \omega_2 X_2(\omega_2)\cos\psi_2 + A\Omega\cos\varphi \\ x_2(0) = x_{2_0} = X_2(\omega_1)\mathrm{sen}\psi_1 + X_2(\omega_2)\mathrm{sen}\psi_2 + B\,\mathrm{sen}\,\varphi \\ \qquad = \omega_1 X_2(\omega_1)\cos\psi_1 + \omega_2 X_2(\omega_2)\cos\psi_2 + B\Omega\cos\varphi \end{cases}$$

A dynamic absorber

For a schema vibrant original to a degree of freedom, consisting of mass m_1 and elastic element of constant k_1 forced by an external forcing type harmonic $F = F_0$ senΩt, is coupled, via a spring k_2, an absorber mass m_2.

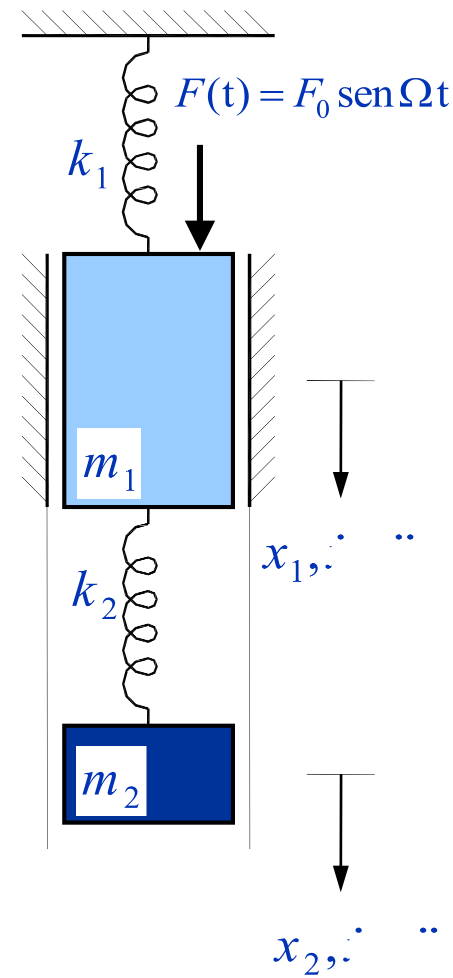
The new scheme is, of course, with two degrees of freedom..



Assorbitore dinamico

The equations of motion are:

$$\begin{cases} F_0 \,\mathrm{sen}\,\Omega t - k_2(x_1 - x_2) - k_1 x_1 = m_1 \ddot{x}_1 \\ -k_2(x_2 - x_1) = m_2 \ddot{x}_2 \end{cases}$$



Introducing the angular frequency of the original system and that of the absorber:

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} \qquad \omega_{22} = \sqrt{\frac{k_2}{m_2}}$$

static displacement of the original system:

$$X_0 = \frac{F_0}{k_1}$$

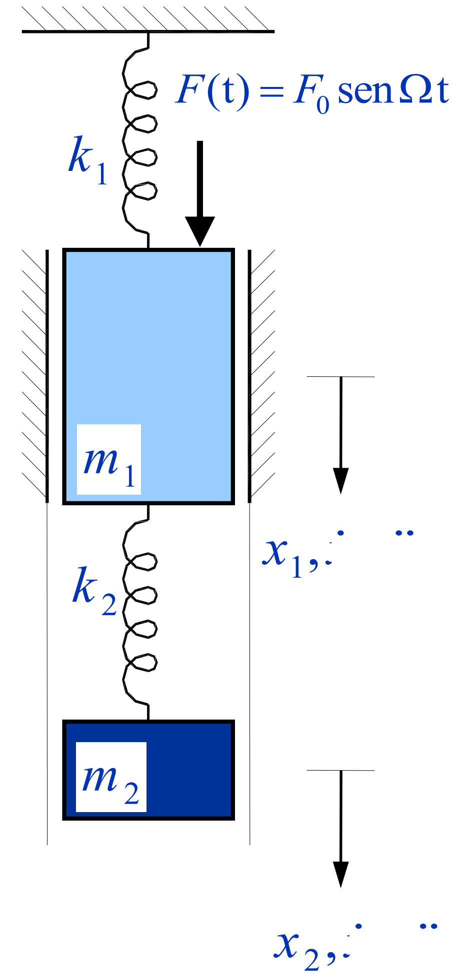
Assorbitore dinamico

We impose the solutions of the forced system:

$$\begin{cases} x_{1part} = A \,\text{sen}\, \Omega t \\ x_{2part} = B \,\text{sen}\, \Omega t \end{cases}$$

The system can be written in the form:

$$\begin{cases} \left[1 + \dfrac{k_2}{k_1} - \left(\dfrac{\Omega}{\omega_{11}}\right)^2\right] X_1 - \left(\dfrac{k_2}{k_1}\right) X_2 = X_0 \\ -X_1 + \left[1 - \left(\dfrac{\Omega}{\omega_{22}}\right)^2\right] X_2 = 0 \end{cases}$$



Assorbitore dinamico

A questo punto il comportamento dinamico del sistema, e perciò la funzione dell'assorbitore, può essere analizzato se si studiano le espressioni delle ampiezze delle due masse riferite allo spostamento statico del sistema originario:

$$\frac{X_1}{X_0} = \frac{1-\left(\dfrac{\Omega}{\omega_{22}}\right)^2}{\left[1+\dfrac{k_2}{k_1}-\left(\dfrac{\Omega}{\omega_{11}}\right)^2\right]\left[1-\left(\dfrac{\Omega}{\omega_{22}}\right)^2\right]-\dfrac{k_2}{k_1}}$$

$$\frac{X_2}{X_0} = \frac{1}{\left[1+\dfrac{k_2}{k_1}-\left(\dfrac{\Omega}{\omega_{11}}\right)^2\right]\left[1-\left(\dfrac{\Omega}{\omega_{22}}\right)^2\right]-\dfrac{k_2}{k_1}}$$

From the first equation it is obvious that the amplitude of the vibrations of the original system X_1 vanishes when the pulsation of forcing Ω coincides with that of the absorber ω_{22}.

Assorbitore dinamico

At this frequency, the amplitude X_2 of the mass m_2 becomes:

$$x_2 = -\frac{k_1}{k_2} X_0 = -\frac{F_0}{k_2}$$

in phase opposition with the forcing.

The principle of the absorber is, in fact, based on his own ability to transmit to the mass m_1, through the spring, a force $k_2 x_2$ equal and opposite to the external F_0. Its mass m_2 depends on the intensity of the exciter force F_0, as the absorber must exert a force equal and opposite to the disturbing force, which in turn depends on the permissible deformation of the spring of the absorber:

$$k_2 X_2 = m_2 \Omega^2 X_2 = -F_0$$

L'assorbitore dinamico può essere usato, per quanto già detto, solo quando la frequenza di disturbo è costante, perchè esso è efficiente solo alla propria frequenza naturale. L'assorbitore dinamico ben si adatta a macchine sincrone ed ad apparecchi funzionanti a corrente alternata a frequenza costante.