DYNAMICS OF SERIAL ROBOTIC MANIPULATORS

NOMENCLATURE AND BASIC DEFINITION

We consider here a mechanical system composed of *r* rigid bodies and denote:

- **$\mathbf{M}_i$** 6x6 inertia dyads of the ith body.
- **Wi** 6 x 6 angular-velocity dyad of the ith body.

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{O} \\ \mathbf{O} & m_i \mathbf{1} \end{bmatrix} \qquad W_i \equiv \begin{bmatrix} \mathbf{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \qquad i = 1, 2 \ldots . r$$

Where:

1 is 3x3 identity matrix

O is 3x3 zero matrix

$\mathbf{\Omega}_i$ is the angular velocity matrix of the *i*th body

$\mathbf{I}_i$ is the inertia matrix of the *i*th body

NOMENCLATURE AND BASIC DEFINITION

Furthermore we define:

- $\mathbf{w}^{w}_{i}$ working wrench exerted on the *i*th body
- $\mathbf{w}^{c}_{i}$ nonworking constraint wrench exerted on the *i*th body
- $\mathbf{t}_{i}$ twist of the *i*th body
- $\boldsymbol{\mu}_{i}$ 6-D momentum screw

$$\mathbf{w}^{W}{}_{i} = \begin{bmatrix} \mathbf{n}_i^W \\ \mathbf{f}_i^W \end{bmatrix} \qquad \mathbf{w}^{c}{}_{i} = \begin{bmatrix} \mathbf{n}_i^C \\ \mathbf{f}_i^C \end{bmatrix} \qquad \mathbf{t}_i = \begin{bmatrix} \boldsymbol{\omega}_i \\ \dot{\boldsymbol{c}}_i \end{bmatrix} \qquad \boldsymbol{\mu}_i = \begin{bmatrix} \mathbf{I}_i \boldsymbol{\omega}_i \\ m_i \dot{\mathbf{c}}_i \end{bmatrix}$$

Where :

- $\mathbf{n}_i$ moment acting on *i*th body
- $\mathbf{f}_i$ force acting on *i*th body

NEWTON-EULER EQUATIONS OF THE ITH RIGID BODY

Clearly, from the definitions of $\mathbf{M}_i$, $\boldsymbol{\mu}_i$ and $\mathbf{t}_i$ we have

$$\boldsymbol{\mu}_i = \mathbf{M}_i \mathbf{t}_i$$

Deriving, we obtain:

$$\dot{\boldsymbol{\mu}}_i = \mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \boldsymbol{\mu}_i = \mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{t}_i$$

Recalling the Newton-Euler equations for a rigid body:

$$\mathbf{I}_i \dot{\boldsymbol{\omega}}_i = -\boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i + \mathbf{n}^{w}{}_i + \mathbf{n}^{c}{}_i$$

$$\mathrm{m_i} \ddot{\mathbf{c}}_\mathrm{i} = \mathbf{f}_i^W + \mathbf{f}_i^C$$

which can be written in compact form using the foregoing definitions, we obtain the Newton-Euler equations of the *i*th body in the form:

$$\mathbf{M}_i \dot{\mathbf{t}}_i = -\mathbf{W}_i \mathbf{M}_i \mathbf{t}_i + \mathbf{w}_i^W + \mathbf{w}_i^C$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

The Euler-Lagrange dynamical equations of a mechanical system are now recalled, as pertaining to serial manipulators. Thus, the mechanical system at hand has n degrees of freedom, its n independent generalized coordinates being the n joint variables, which are stored in the n-dimensional vector $\boldsymbol{\theta}$. We thus have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right) - \left(\frac{\partial T}{\partial \boldsymbol{\theta}}\right) = \phi$$

where T is a scalar function denoting the kinetic energy of the system and ϕ is the n-dimensional vector of generalized force.

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

We can decompose ϕ into two parts, ϕ_p and ϕ_n, the former arising from V and termed the conservative force of the system; the latter is the nonconservative force. That is:

$$\phi_p \equiv -\frac{\partial V}{\partial \boldsymbol{\theta}}$$

The above Euler-Lagrange equations thus becoming:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial \mathrm{L}}{\partial \boldsymbol{\theta}} = \phi_{\mathrm{n}}$$

Where L is the Lagrangian of the system, defined as:

$$L \equiv T - V$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

Moreover, the kinetic energy of the system is given by the sum of all the kinetic energies of r links. The kinetic energy of the rigid body in term of generalized variables is:

$$T = \sum_{i}^{r} T_i = \sum_{i}^{r} \frac{1}{2} \mathbf{t}_i^T \mathbf{M}_i \mathbf{t}_i$$

whereas the vector of nonconservative generalized forces is given by: $$\phi_n \equiv \frac{\partial \Pi^A}{\partial \dot{\boldsymbol{\theta}}} - \frac{\partial \Pi^D}{\partial \dot{\boldsymbol{\theta}}}$$

As denotes $\prod^{A}$ the active power and the $\prod^{D}$ power dissipated in the system

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

The wrench $\mathbf{w}_i^E$ is decomposed into two parts: $\mathbf{w}_i^A$ active wrench supplied by the actuators and $\mathbf{w}_i^D$ dissipative wrench arises from viscous and Coulomb friction

Thus, the power supplied to the *i*th link is readily computed as:

$$\Pi_i{}^A = (\mathbf{w}_i^A)^T \mathbf{t}_i \qquad \Pi_i{}^D = (\mathbf{w}_i^D)^T \mathbf{t}_i$$

Similar to the kinetic energy, then, the power supplied to the overall system is simply the sum of the individual powers supplied to each link, and expressed as:

$$\Pi^A = \sum_1^r \Pi^A{}_i \qquad \Pi^D = \sum_1^r \Pi^D{}_i$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

Further definitions are now introduced:

- $\mathbf{t}$ manipulator twist
- $\boldsymbol{\mu}$ manipulator momentum
- $\mathbf{w}^C$ manipulator constraint wrench
- $\mathbf{w}^A$ manipulator active wrench
- $\mathbf{w}^D$ manipulator dissipative wrench

$$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{t}_n \end{bmatrix} \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\mu}_n \end{bmatrix} \quad \mathbf{w}^C = \begin{bmatrix} \mathbf{w}_1^C \\ \mathbf{w}_2^C \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{w}_n^C \end{bmatrix} \quad \mathbf{w}^A = \begin{bmatrix} \mathbf{w}_1^A \\ \mathbf{w}_2^A \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{w}_n^A \end{bmatrix} \quad \mathbf{w}^D = \begin{bmatrix} \mathbf{w}_1^D \\ \mathbf{w}_2^D \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{w}_n^D \end{bmatrix}$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

Additionally, the $6n$x$6n$ matrices of manipulator mass **M** and manipulator angular velocity **W** are also introduced below:

$$\mathbf{M} = \text{diag}\left(\mathbf{M}_1, \mathbf{M}_2, \ldots \ldots, \mathbf{M}_n\right)$$

$$\mathbf{W} = \text{diag}\left(\mathbf{W}_1, \mathbf{W}_2, \ldots \ldots, \mathbf{W}_n\right)$$

From this definitions we have:

$$\boldsymbol{\mu} = \mathbf{Mt}$$

$$\dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{WMt}$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

With the foregoing definitions, then, the kinetic energy of the manipulator takes on a simple form, namely,

$$T = \frac{1}{2}\mathbf{t}^T\mathbf{M}\mathbf{t} = \frac{1}{2}\mathbf{t}^T\boldsymbol{\mu}$$

Which is a quadratic form in the system twist. Since the twist, on the other hand, is a linear function of the vector $\dot{\boldsymbol{\theta}}$ of joint rates, the kinetic energy turns out to be a quadratic form in the vector of joint rates.

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

Moreover, we will assume that this form is homogeneous in $\dot{\boldsymbol{\theta}}$:

$$T = \frac{1}{2}\dot{\boldsymbol{\theta}}^T \mathbf{I}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

Notice that the above assumption implies that the base of the robot is fixed to an inertial base, and hence, when all joints are locked, the kinetic energy of the robot vanishes.

It is apparent that: $\mathbf{I}(\boldsymbol{\theta}) = \frac{\partial^2}{\partial \dot{\boldsymbol{\theta}}^2}(T)$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

Furthermore, the Euler-Lagrange equations can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial T}{\partial \boldsymbol{\theta}} + \frac{\partial V}{\partial \boldsymbol{\theta}} = \phi_n$$

The partial derivatives appearing in the foregoing equation take the forms derived below:

$$\frac{dT}{d\dot{\boldsymbol{\theta}}} = \mathbf{I}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

and hence: $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right) = \mathbf{I}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

We express the kinetic energy in the form:

$$T = \frac{1}{2}\mathbf{p}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)^T \dot{\boldsymbol{\theta}}$$

Where $\mathbf{p}(\theta, \dot{\theta})$is the generalized momentum of the manipulator, defined as:

$$\mathbf{p}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) = \mathbf{I}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

Hence:

$$\frac{\partial T}{\partial \boldsymbol{\theta}} = \frac{1}{2}\left(\frac{\partial \mathbf{p}}{\partial \boldsymbol{\theta}}\right)^T \dot{\boldsymbol{\theta}}$$

THE EULER-LAGRANGE EQUATIONS OF SERIAL MANIPULATORS

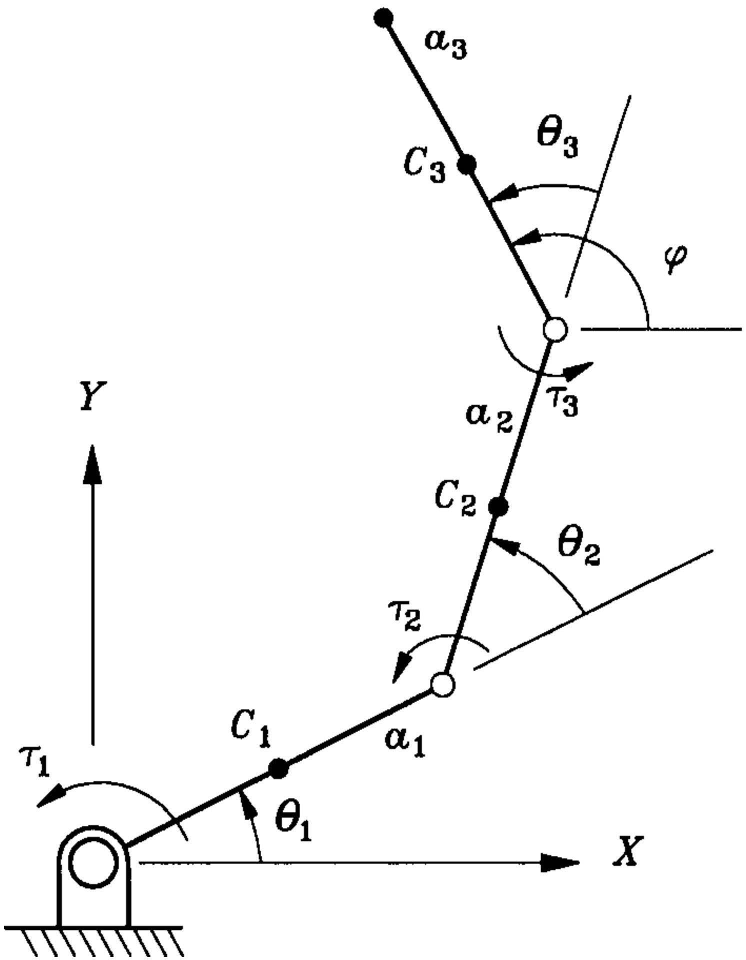
The Euler-Lagrange equations thus taking on the alternative form:

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \frac{1}{2}\left[\frac{\partial(\mathbf{I}\dot{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]^T \dot{\boldsymbol{\theta}} + \frac{\partial V}{\partial\boldsymbol{\theta}} = \phi_n$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

Consider the manipulator of Figure with links designed so that their mass centers C_1, C_2, C_3 are located at the midpoints of segments O_1O_2, O_2O_3, and O_3P,respectively. Moreover, the *i*th link has a mass m_i and a centroidal moment of inertia in a direction normal to the plane of motion I_i, while the joints are actuated by motors delivering torques τ_1, τ_2, and τ_3 the lubricant of the joints producing dissipative torques that we will neglect in this model. Under the assumption that gravity acts in the direction of —Y, find the associated Euler-Lagrange equations.

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT



SOLUTION:

All vectors introduced below are 2-D, the scalar angular velocities of the links *ωi*, for *i* =1, 2, 3, being:

$$\omega_1 = \dot{\theta}_1$$

$$\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$$

$$\omega_3 = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

Moreover, the velocities of the mass centers are:

$$\dot{\mathbf{c}}_1 = \frac{1}{2}\dot{\theta}_1 \mathbf{E}\mathbf{a}_1$$

$$\dot{\mathbf{c}}_2 = \dot{\theta}_1 \mathbf{E}\mathbf{a}_1 + \frac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\mathbf{E}\mathbf{a}_2$$

$$\dot{\mathbf{c}}_3 = \dot{\theta}_1 \mathbf{E}\mathbf{a}_1 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)\mathbf{E}\mathbf{a}_2 + \frac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)\mathbf{E}\mathbf{a}_3$$

The kinetic energy then becoming:

$$T = \frac{1}{2}\sum_{1}^{3}\left(m_i \|\dot{\mathbf{c}}_i\|^2 + I_i {\omega^2}_i\right)$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

The squared magnitudes of the mass-center velocities are now computed using the expressions derived above. After simplifications, these yield

$$\|\dot{\mathbf{c}}_1\|^2 = \frac{1}{4}{a_1}^2{\dot{\theta}_1}^2$$

$$\|\dot{\mathbf{c}}_2\|^2 = {a_1}^2{\dot{\theta}_1}^2 + \frac{1}{4}{a_2}^2\left({\dot{\theta}_1}^2 + 2\dot{\theta}_1\dot{\theta}_2 + {\dot{\theta}_2}^2\right) + a_1 a_2 \cos\theta_2\left({\dot{\theta}_1}^2 + \dot{\theta}_1\dot{\theta}_2\right)$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

The kinetic energy of the whole manipulator thus becomes:

$$T = \frac{1}{2}(I_{11}\dot{\theta}_1{}^2 + 2I_{12}\dot{\theta}_1\dot{\theta}_2 + 2I_{23}\dot{\theta}_1\dot{\theta}_3 + I_{22}\dot{\theta}_2{}^2 + 2I_{33}\dot{\theta}_3{}^2)$$

with coefficients I_{ij}, for $i = 1, 2, 3$, and $j = i$ to 3 being the distinct entries of the 3 × 3 matrix of generalized inertia of the system. These entries are given below:

$$I_{11} = I_1 + I_2 + I_3 + \frac{1}{4}m_1 a_1{}^2 + m_2\left(a_1{}^2 + \frac{1}{4}a_2{}^2 + a_1 a_2 c_2\right) + m_3\left(a_1{}^2 + a_2{}^2 + \frac{1}{4}a_3{}^2 + 2a_1 a_3 c_2 + a_1 a_3 c_{23} + a_2 a_3 c_3\right)$$

$$I_{12} = I_2 + I_3 + \frac{1}{2}\left[m_2\left(\frac{1}{2}a_2^2 + a_1 a_2 c_2\right)\right] + \left[m_3\left(2a_2^2 + \frac{1}{2}a_3^2 + 2a_1 a_2 c_2 + a_1 a_3 c_{23} + 2a_2 a_3 c_3\right)\right]$$

$$I_{13} = I_3 + \frac{1}{2}\left(\frac{1}{2}a_3^2 + a_1 a_3 c_{23} + a_2 a_3 c_3\right)$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

$$I_{22} = I_2 + I_3 + \frac{1}{4}m_2a_2^2 + m_3\left(a_2^2 + \frac{1}{4}a_3^2 + a_2a_3c_3\right)$$

$$I_{23} = I_3 + \frac{1}{2}m_3(a_3^2 + a_2a_3c_3)$$

$$I_{33} = I_3 + \frac{1}{4}m_3a_3^2$$

where: $c_i = \cos\theta_i$ e $c_{ij} = \cos(\theta_i + \theta_j)$

Furthermore, the potential energy of the manipulator is computed as the sum of the individual link potential energies :

$$V = \frac{1}{2}m_1ga_1sen\theta_1 + m_2g\left[a_1sen\theta_1 + \frac{1}{2}a_2sen(\theta_1 + \theta_2)\right] + m_3g\left[a_1sen\theta_1 + a_2sen\,(\theta_1 + \theta_2) + \frac{1}{2}a_3sen\,(\theta_1 + \theta_2 + \theta_3)\right]$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

While the total power delivered to the manipulator takes the form:

$$\Pi = \tau_1\dot{\theta}_1 + \tau_2\dot{\theta}_2 + \tau_3\dot{\theta}_3$$

We now proceed to compute the various terms:

$$\dot{I}_{11} = -m_2a_1a_2s_2\dot{\theta}_2 - m_3[2a_1a_2s_2\dot{\theta}_2 + a_1a_3s_{23}(\dot{\theta}_2 + \dot{\theta}_3) + a_2a_3s_3\dot{\theta}_3]$$

$$\dot{I}_{22} = \frac{1}{2}\{-m_2a_1a_2s_2\dot{\theta}_2 - m_3[2a_1a_2s_2\dot{\theta}_2 + a_1a_3s_{23}(\dot{\theta}_2 + \dot{\theta}_3) + a_2a_3s_3\dot{\theta}_3]\}$$

$$\dot{I}_{12} = -\frac{1}{2}m_3[a_1a_3s_{23}(\dot{\theta}_2 + \dot{\theta}_3) + a_2a_3s_3\dot{\theta}_3]$$

$$\dot{I}_{22} = -m_3a_2a_3s_3\dot{\theta}_3$$

$$\dot{I}_{23} = -\frac{1}{2}m_3a_2a_3s_3\dot{\theta}_3$$

$$\dot{I}_{33} = 0$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

We have then:

$$\dot{\mathbf{I}}\dot{\boldsymbol{\theta}} = \mathbf{h} = \begin{bmatrix} \dot{I}_{11}\dot{\theta}_1 + \dot{I}_{12}\dot{\theta}_2 + \dot{I}_{13}\dot{\theta}_3 \\ \dot{I}_{12}\dot{\theta}_1 + \dot{I}_{22}\dot{\theta}_2 + \dot{I}_{23}\dot{\theta}_3 \\ \dot{I}_{13}\dot{\theta}_1 + \dot{I}_{23}\dot{\theta}_2 + \dot{I}_{33}\dot{\theta}_3 \end{bmatrix}$$

whose components are readily calculated as:

$$h_1 = -[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1\dot{\theta}_2 - m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_1\dot{\theta}_3 - \frac{1}{2}[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_2^2 - m_3(a_1 a_3 s_{23})\dot{\theta}_2\dot{\theta}_3 - \frac{1}{2}m_3(a_1 a_3 s_{23})\dot{\theta}_3^2$$

$$h_2 = \frac{1}{2}[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}m_3(a_1 a_3 s_3 + a_2 a_3 s_3)\dot{\theta}_1\dot{\theta}_3 - m_3 a_2 a_3 s_3\dot{\theta}_2\dot{\theta}_3 - \frac{1}{2}m_3 a_2 a_3 s_3\theta_3^2$$

$$h_3 = -\frac{1}{2}m_3 a_1 a_3 s_{23}\dot{\theta}_1\dot{\theta}_3 - \frac{1}{2}m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_1\dot{\theta}_3 - \frac{1}{2}m_3 a_2 a_3 s_3\dot{\theta}_2\dot{\theta}_3$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

$\frac{\partial \mathbf{I}\dot{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}} = I'$ its entries being denoted by I'_{ij}. This matrix, in component form, is given by :

$$\mathbf{I}' = \begin{bmatrix} 0 & I_{11,2}\dot{\theta}_1 + I_{12,2}\dot{\theta}_2 + I_{13,2}\dot{\theta}_3 & I_{11,3}\dot{\theta}_1 + I_{12,3}\dot{\theta}_2 + I_{13,3}\dot{\theta}_3 \\ 0 & I_{12,2}\dot{\theta}_1 + I_{22,2}\dot{\theta}_2 + I_{13,2}\dot{\theta}_3 & I_{12,3}\dot{\theta}_1 + I_{22,3}\dot{\theta}_2 + I_{23,3}\dot{\theta}_3 \\ 0 & I_{13,2}\dot{\theta}_1 + I_{23,2}\dot{\theta}_2 + I_{33,2}\dot{\theta}_3 & I_{13,3}\dot{\theta}_1 + I_{23,3}\dot{\theta}_2 + I_{33,3}\dot{\theta}_3 \end{bmatrix}$$

with the shorthand notation $I_{ij,k}$ indicating the partial derivative of I_{ij} with respect to θ_k. As the reader can verify, these entries are given as:

$$I'_{11} = 0$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

$$I'_{12} = -[m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_1 - \frac{1}{2}[m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_2 - \frac{1}{2}m_3a_1a_3s_{23}\dot{\theta}_3$$

$$I'_{13} = -m_3(a_1a_3s_{23} + a_2a_3s_3)\dot{\theta}_1 - \frac{1}{2}m_3(a_1a_3s_{23} + 2a_2a_3s_3)\dot{\theta}_2 - \frac{1}{2}m_3(a_1a_3s_{23} + a_2a_3s_3)\dot{\theta}_3$$

$$I'_{21} = 0$$

$$I'_{22} = -\frac{1}{2}m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_1$$

$$I'_{23} = -\frac{1}{2}m_3(a_1a_2s_{23} + 2a_2a_3s_3)\dot{\theta}_1 - m_3a_2a_3s_3\dot{\theta}_2 - \frac{1}{2}m_3a_2a_3s_3\dot{\theta}_3$$

$$I'_{31} = 0$$

$$I'_{32} = -\frac{1}{2}m_3a_1a_3s_{23}\dot{\theta}_1$$

$$I'_{33} = -\frac{1}{2}m_3(a_1a_3s_{23} + a_2a_3s_3)\dot{\theta}_1 - \frac{1}{2}m_3a_2a_3s_3\dot{\theta}_2$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

Now, we define the 3-dimensional vector $\boldsymbol{\gamma}$ below:

$$\boldsymbol{\gamma} = \left[\frac{\partial(\mathbf{I}\,\dot{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]^T \dot{\boldsymbol{\theta}}$$

its three components being:

$$\gamma_1 = 0$$

$$\gamma_2 = -[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1^2 - [m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1^2\dot{\theta}_2^2 - m_3 a_1 a_3 s_{23}\theta_1^2\theta_3^2$$

$$\gamma_3 = -m_3(a_1 a_2 s_{23} + a_2 a_3 s_3)\dot{\theta}_1^2 - m_3(a_1 a_3 s_{23} + 2a_2 a_3 s_{23} + 2a_2 a_3 s_3)\dot{\theta}_1^2\dot{\theta}_2^2 - m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_1\dot{\theta}_3 - m_3 a_2 a_3 s_3\dot{\theta}_3^2 - m_3 a_2 a_3 s_3\dot{\theta}_2\dot{\theta}_3$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

We now turn to the computation of the partial derivatives of the potential energy:

$$\frac{\partial V}{\partial \theta_1} = \frac{1}{2} m_1 g a_1 c_1 + m_2 g \left(a_1 c_1 + \frac{1}{2} a_2 c_{12} \right) + m_3 g \left(a_1 c_1 + a_2 c_{12} + \frac{1}{2} a_2 c_{123} \right)$$

$$\frac{\partial V}{\partial \theta_2} = \frac{1}{2} m_2 g a_2 + m_3 g \left(a_2 c_{12} + \frac{1}{2} a_3 c_{123} \right)$$

$$\frac{\partial V}{\partial \theta_3} = \frac{1}{2} m_3 g a_3 c_{123}$$

EULER-LAGRANGE EQUATIONS OF A PLANAR ROBOT

The Euler-Lagrange equations thus reduce to:

$$I_{11}\ddot{\theta}_1 + I_{12}\ddot{\theta}_2 + I_{13}\ddot{\theta}_3 + h_1 - \frac{1}{2}\gamma_1 + \frac{1}{2}m_1 g a_1 c_1 + m_2 g\left(a_1 c_1 + \frac{1}{2}a_2 c_{12}\right) + m_3 g\left(a_1 c_1 + a_2 c_{12} + \frac{1}{2}a_2 c_{123}\right) = \tau_1$$

$$I_{12}\ddot{\theta}_1 + I_{12}\ddot{\theta}_2 + I_{23}\ddot{\theta}_3 + h_2 - \frac{1}{2}\gamma_2 + \frac{1}{2}m_2 g a_2 + m_3 g\left(a_2 c_{12} + \frac{1}{2}a_3 c_{123}\right) = \tau_2$$

$$I_{13}\ddot{\theta}_1 + I_{23}\ddot{\theta}_2 + I_{33}\ddot{\theta}_3 + h_3 - \frac{1}{2}\gamma_3 + \frac{1}{2}m_3 g a_3 c_{123} = \tau_3$$