THE NATURAL ORTHOGONAL COMPLEMENT (N.O.C.) IN ROBOT DYNAMICS

THE NATURAL ORTHOGONAL COMPLEMENT

The Newton-Euler equations obtained from the study of a serial manipulator does not constitute a mathematical model because they can not use recursive relations.

We want to determine a model that describes the status assumed by the system as a function of the generalized external forces applied.

We want to derive the mathematical model for a serial manipulator in function of its kinematic structure and its inertial properties.

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We recall the Newton-Euler equations of the ith body in 6-D form:

$$\mathbf{M}_i \dot{\mathbf{t}}_i = -\mathbf{W}_i \mathbf{M}_i \mathbf{t}_i + \mathbf{w}_i^W + \mathbf{w}_i^C \quad i=1,2,\ldots,n$$

Where: $$\mathbf{t} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ . \\ . \\ . \\ \mathbf{t}_n \end{bmatrix} \quad \mathbf{w}^C = \begin{bmatrix} \mathbf{w}^C{}_1 \\ \mathbf{w}^C{}_2 \\ . \\ . \\ . \\ \mathbf{w}^C{}_n \end{bmatrix}$$

$$\mathbf{M} = \mathrm{diag}\,(\mathbf{M}_1, \mathbf{M}_2, \ldots \ldots, \mathbf{M}_n) \; \mathbf{W} = \mathrm{diag}\,(\mathbf{W}_1, \mathbf{W}_2, \ldots \ldots, \mathbf{W}_n)$$

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If w^w is decomposed into its active ($\mathbf{w}^A$), gravitational($\mathbf{w}^G$)and dissipative($\mathbf{w}^D$) parts, the foregoing equation takes the form:

$$\mathbf{M\dot{t}} = -\mathbf{WMt} + \mathbf{w}_C + \mathbf{w}_A + \mathbf{w}_G + \mathbf{w}_D$$

Since the gravity acts at the mass centre of a body, the gravity wrench $\mathbf{w}_i^G$ acting on the ith link takes the form:

$$\boldsymbol{w}_i^G = \begin{bmatrix} \mathbf{0} \\ \mathrm{m_i}\,\mathbf{g} \end{bmatrix}$$

THE NATURAL ORTHOGONAL COMPLEMENT

$$\mathbf{M}\dot{\mathbf{t}} = -\mathbf{W}\mathbf{M}\mathbf{t} + \mathbf{w}_{\mathrm{C}} + \mathbf{w}_{\mathrm{A}} + \mathbf{w}_{\mathrm{G}} + \mathbf{w}_{\mathrm{D}}$$

The mathematical model displayed in this equation represents the uncoupled Newton-Euler equations of the overall manipulator.

The following steps of this derivation consists in representing the coupling between every two consecutive links as a linear homogeneous system of algebraic equations on the link twist.

We note that all kinematic pairs allow a relative 1 dof motion between the coupled bodies.

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We can expres the kinematic constraints of the system in linear homogeneous form in the 6n-D vector of manipulator twist, namely

$$\mathbf{Kt} = \mathbf{0}$$

Furthermore, since the nonworking constraint wrench $\mathbf{w}^C$ produces no work on the manipulator, its sole function being to keep the links together, the power developed by this wrench is zero:

$$\mathbf{t}^T\mathbf{w}^C = \mathbf{0}$$

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If we do the transpose of equation **Kt**=0 and multiply by a vector **λ** 6n-D, we obtain:

$$\mathbf{t}^T\mathbf{K}^T\boldsymbol{\lambda} = 0$$

it is apparent that $\mathbf{w}^C$ takes the form

$$\mathbf{w}^c = \mathbf{K}^T\boldsymbol{\lambda}$$

Then we represent the twist as a linear transformation of the independent generalized speeds

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}}$$

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Substituting this into the equation **KT=0** we get:

$$\mathbf{KT}\dot{\boldsymbol{\theta}} = \mathbf{0}$$

Since the degree of freedom of the manipulator is n, the n $\{\dot{\theta}\}_1^n$ generalized velocity can be assigned arbitrarily.

The above equation must still be verified: for this to happen it is necessary that **KT=0**

Therefore, we can say that **T** is the orthogonal complement of **K.**

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We replace the equation of Euler-Neton the following relationship, which originated from the foregoing considerations

$$\dot{\mathbf{t}} = \mathbf{T}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}$$

Premultiplying for the matrix $\mathbf{T}^{\mathrm{T}}$, we reduce the system of n independent equations free from the forces of constraint, known as the Euler-Lagrange equations:

$$\boldsymbol{I}\,\ddot{\theta} = -\mathbf{T}^{\mathrm{T}}\left(\mathbf{M}\dot{\mathbf{T}} - \mathbf{WMT}\right)\dot{\theta} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{w}^{\mathrm{A}} + \mathbf{w}^{\mathrm{D}} + \mathbf{w}^{\mathrm{G}}\right)$$

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$$\mathbf{I}\,\ddot{\boldsymbol{\theta}} = -\mathbf{T}^{\mathrm{T}}\left(\mathbf{M}\dot{\mathbf{T}} - \mathbf{WMT}\right)\dot{\boldsymbol{\theta}} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{w}^{\mathrm{A}} + \mathbf{w}^{\mathrm{D}} + \mathbf{w}^{\mathrm{G}}\right)$$

Where **I** is the positive definite n x n generalized inertia matrix of the manipulator and is defined as

$$\mathbf{I} = \mathbf{T}^{\mathrm{T}}\mathbf{MT}$$

we let τ and δ denote the n-dimensional vectors of active and dissipative generalized force.

$$\boldsymbol{\tau} = \mathbf{T}^{\mathrm{T}}\mathbf{w}^{\mathrm{A}} \qquad \boldsymbol{\delta} = \mathbf{T}^{\mathrm{T}}\mathbf{w}^{\mathrm{D}} \qquad \boldsymbol{\gamma} = \mathbf{T}^{\mathrm{T}}\mathbf{w}^{\mathrm{G}}$$

Moreover, we let $\mathbf{C}(\theta,\dot{\theta})\,\dot{\theta}$ be the n-dimensional vector of quadratic terms of inertia force.

$$\mathbf{C}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) = -\mathbf{T}^{\mathrm{T}}(\mathbf{MT} + \mathbf{WMT})$$

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Thus, the Euler-Lagrange equations of the system take on the form:

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{\tau} + \boldsymbol{\gamma} + \boldsymbol{\delta}$$

If, moreover, a static wrench $\mathbf{w}^{W}$ acts onto the EE, then its effect onto the above model is taken into account by adding a term $\mathbf{J}^{T}\mathbf{w}^{W}$

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} = \mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}} + \boldsymbol{\tau} + \boldsymbol{\gamma} + \boldsymbol{\delta} + \mathbf{J}^{T}\mathbf{w}^{W}$$

CONSTRAINT EQUATIONS

If $\mathbf{E}_i$ is defined as the cross-product matrix of vector $\mathbf{e}_i$, then the angular velocities of two successive links obey a simple relation, namely.

$$\mathbf{E}_i(\omega_i - \omega_{i-1}) = 0$$

Considering a rotational joint:

$$\dot{\mathbf{c}}_i = \dot{\mathbf{c}}_{i-1} + \omega_{i-1} \times \boldsymbol{\delta}_{i-1} + \omega_i \times \boldsymbol{\rho}_i$$

$$\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i \omega_i + \mathbf{D}_{i-1} \omega_{i-1} = 0$$

where $\mathbf{D}_i$ and $\mathbf{R}_i$ are defined as the cross-product matrices of vectors $\boldsymbol{\delta}_i$ and $\boldsymbol{\rho}_i$

CONSTRAINT EQUATIONS

In particular, if the first link is inertial, the above equations become

$$\mathbf{E_1}\omega_1 = 0 \qquad \mathbf{c}_i + \mathbf{R}_1\omega_1 = 0$$

Expressing these equations in terms of link twists we have:

$$\mathbf{K}_{11}\mathbf{t}_1 = 0$$
$$\mathbf{K}_i, \mathbf{K}_{i-1}\mathbf{t}_{i-1} + \mathbf{K}_{ii}\mathbf{t}_i = 0$$

Where:

$$K_{11} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{R}_1 & \mathbf{1} \end{bmatrix} \quad \mathbf{K}_{i,i-1} = \begin{bmatrix} -\mathbf{E}_i & \mathbf{0} \\ \mathbf{D}_{i-1} & -\mathbf{1} \end{bmatrix} \quad \mathbf{K}_{ii} = \begin{bmatrix} \mathbf{E}_i & \mathbf{0} \\ \mathbf{R}_i & \mathbf{1} \end{bmatrix}$$

CONSTRAINT EQUATIONS

From the foregoing equations the **K** matrix, appering in **Kt=0** , takes on the form:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{0}_6 & \mathbf{0}_6 & \cdots & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 & \mathbf{0}_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \cdots & \mathbf{K}_{n-1,n-1} & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \cdots & \mathbf{K}_{n,n-1} & \mathbf{K}_{n,n} \end{bmatrix}$$

With $\mathbf{0}_6$ denoting the 6 x 6 zero matrix.

CONSTRAINT EQUATIONS

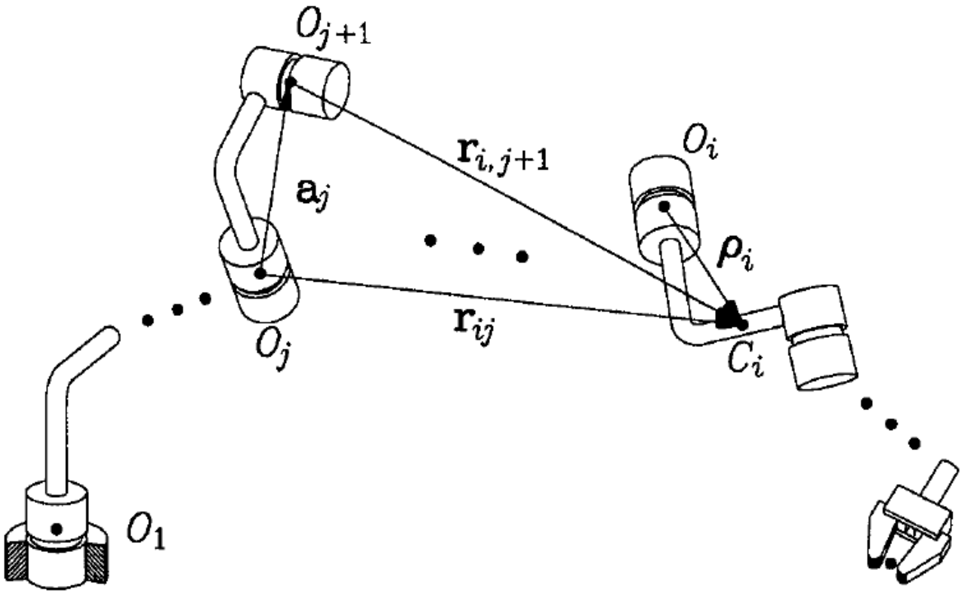
Further, the link-twists are expressed as linear combinations of the joint-rate vector $\dot{\boldsymbol{\theta}}$. To this end, we define the 6 x n partial Jacobian $\mathbf{J}_i$ as the matrix mapping the joint-rate vector $\boldsymbol{\theta}$ into the twist $\mathbf{t}_i$

$$\mathbf{J}_i \dot{\boldsymbol{\theta}} = \mathbf{t}_i$$

whose jth column, t_{ij}, is given, for i, j = 1, 2, . . . , n, by

$$\mathbf{t}_{ij} = \begin{cases} \begin{bmatrix} \mathbf{e}_j \\ \mathbf{e}_j \times \mathbf{r}_{ij} \end{bmatrix} & j \leq i \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} & \end{cases}$$

CONSTRAINT EQUATIONS



With $\mathbf{r}_{ij}$ illustrated and defined as:

$$\mathbf{r}_{ij} = \begin{cases} \mathbf{a}_j + \mathbf{a}_{j+1} + \mathbf{a}_{i-1} + \boldsymbol{\rho}_i & j < i \\ \boldsymbol{\rho}_i & j = 1 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

CONSTRAINT EQUATIONS

We can thus readily express the twist $\mathbf{t}_i$ of the ith link as a linear combination of the first i joint rates, namely.

$$\mathbf{t}_i = \dot{\theta}_1 \mathbf{t}_{i1} + \dot{\theta}_2 \mathbf{t}_{i2} + \dot{\cdots} + \dot{\theta}_i \mathbf{t}_{ii}$$

and hence, matrix $\mathbf{T}$ takes the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{t}_{21} & \mathbf{t}_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{t}_{n1} & \mathbf{t}_{n2} & \cdots & \mathbf{t}_{nn} \end{bmatrix}$$

CONSTRAINT EQUATIONS

The kinematic constraint equations on the twists, for the case in which the ith joint is prismatic, are obtained in the same way.

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1}$$

$$\dot{\mathbf{c}}_{\mathrm{i}} = \dot{\mathbf{c}_{\mathrm{i}-1}} + \boldsymbol{\omega}_{\mathrm{i}-1} \times (\boldsymbol{\delta}_{\mathrm{i}-1} + \mathbf{p}_{\mathrm{i}} + b_{\mathrm{i}}\mathbf{e}_{\mathrm{i}}) + \dot{b_{\mathrm{i}}}\mathbf{e}_{\mathrm{i}}$$

We introduce one further definition:

$$\mathbf{R}_{\mathrm{i}}' = \mathbf{D}_{\mathrm{i}-1}' + \mathbf{R}_{\mathrm{i}}$$

where $\mathbf{D}_{i-1}$ is the cross-product matrix of vector $\boldsymbol{\delta}_{i-1}$, while $\mathbf{R}_i$ is the cross-product matrix of $\boldsymbol{\varrho}_i+b_i\mathbf{e}_i$

Hence the above equation can be rewritten as:

$$\dot{\mathbf{c}}_{\mathrm{i}} - \dot{\mathbf{c}_{\mathrm{i}-1}} + \mathbf{R}_{\mathrm{i}}'\boldsymbol{\omega}_{\mathrm{i}} - \dot{b_{\mathrm{i}}}\mathbf{e}_{\mathrm{i}} = 0$$

CONSTRAINT EQUATIONS

Multiplying both sides of the equation by $\mathbf{E}_i$ the term $\dot{b}_i$ vanishes, we thus obtain:

$$\mathbf{E}_i(\dot{\mathbf{c}}_i - \dot{\mathbf{c}}_{i-1} + \mathbf{R}_i'\boldsymbol{\omega}_i) = \mathbf{0}$$

The above equations can be grouped into the system linear and homogeneous in the unknown twist.

$$\mathbf{K}_{i,i-1}'\mathbf{t}_{i-1} + \mathbf{K}_{ii}'\mathbf{t}_i = \mathbf{0}$$

the associated matrices being defined below

$$\mathbf{K}_{i,i-1}' = \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{E}_i \end{bmatrix} \qquad \mathbf{K}_{i,i}' = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{E}_i\mathbf{R}_i' & \mathbf{E}_i \end{bmatrix}$$

CONSTRAINT EQUATIONS

If the first joint is prismatic, then the corresponding constraint equation takes on the form:

$$\mathbf{K}'_{11}\mathbf{t}_1 = \mathbf{0}$$

Where: $$\mathbf{K}'_{11} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_\mathrm{i} \end{bmatrix}$$

Furthermore, if the kth pair is prismatic $1 \leq \mathrm{k} \leq \mathrm{i}$ then the twist **t**i of the ith link changes to

$$\mathbf{t}_\mathrm{i} = \dot{\theta}_1\mathbf{t}_{\mathrm{i}1} + \cdots + \dot{b}_k\mathbf{t}'_{\mathrm{ik}} + \cdots + \dot{\theta}_\mathrm{i}\mathbf{t}_{\mathrm{ii}}$$

Where: $$\mathbf{t}'_{\mathrm{ik}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_\mathrm{k} \end{bmatrix}$$

CONSTRAINT EQUATIONS

If the pair is a revolute joint $\dot{\mathbf{t}}_{ij}$ is obtained by deriving $\mathbf{t}_{ij}$ from foregoing equations:

$$\dot{\boldsymbol{t}}_{ij} = \begin{cases} \begin{bmatrix} \boldsymbol{\omega}_j \times \mathbf{e}_j \\ (\boldsymbol{\omega}_j \times \mathbf{e}_j) \times \mathbf{e}_j \times \mathbf{e}_{ij} \end{bmatrix} & \text{se } j \le i \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} & \text{otherwise} \end{cases}$$

Where:

$$\dot{\mathbf{r}}_{ij} = \boldsymbol{\omega}_j \times \mathbf{a}_j + \cdots + \boldsymbol{\omega}_{j-1} \times \mathbf{a}_{j-1} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i$$

If the pair is prismatic the time-rate of change t_{ik}' becomes:

$$\dot{\mathbf{t}}'_{ik} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}_k \times \mathbf{e}_k \end{bmatrix}$$

NONINERTIAL BASE LINK

Noninertial bases occur in space applications. A noninertial base can be readily handled with the use of the natural orthogonal complement.

Since the base is free of attachments to an inertial frame,we have to add its six degrees of freedom (dof) to the *n* dof of the rest of the manipulator.

In particular, **t**, $\mathbf{w}^{C}$, $\mathbf{w}^{A}$, and $\mathbf{w}^{D}$ now become 6(n + 1)-D vectors and they take the form:

$$\mathbf{t} = [\mathbf{t}_0^{\mathrm{T}} \quad \mathbf{t}_1^{\mathrm{T}} \quad \cdots \quad \mathbf{t}_{\mathrm{n}}^{\mathrm{T}}]^{\mathrm{T}} \qquad \dot{\boldsymbol{\theta}} = [\mathbf{t}_0^{\mathrm{T}} \quad \dot{\theta}_1 \quad \cdots \quad \dot{\theta}_n]$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{T}' \end{bmatrix} \qquad \mathbf{V} = [\mathbf{t}_0^{\mathrm{T}} \quad \dot{\theta}_1 \quad \cdots \quad \dot{\theta}_n]^{\mathrm{T}}$$