N.O.C. for PLANAR MANIPULATORS

PLANAR MANIPULATORS

The application of the natural orthogonal complement to planar manipulators is straightforward.

Here, we assume that the manipulator at hand is composed of *n* links coupled by *n* joints of the revolute or the prismatic type.

Moreover, for conciseness, we assume that the first link, labeled the base, is fixed to an inertial frame.

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We define the twist of the *ith* link and the wrench acting on it as 3-dimensional arrays, namely:

$$\mathbf{t}_i = \begin{bmatrix} \omega_i \\ \dot{\mathbf{c}}_i \end{bmatrix} \quad \mathbf{w}_i = \begin{bmatrix} n_i \\ \mathbf{f}_i \end{bmatrix}$$

where :

- ω_i is the scalar angular velocity of this link;
- $\dot{\mathbf{c}}_i$ is the 2-D velocity of its mass center C_i;
- n_i is the scalar moment acting on the link;
- $\mathbf{f}_i$ is the 2-D force acting at C_i.

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Moreover, the inertia dyad is now a 3 x 3 matrix:

$$\mathbf{M}_\mathrm{i} = \begin{bmatrix} I_i & \mathbf{0}^\mathrm{T} \\ \mathbf{0} & m_i\mathbf{1} \end{bmatrix}$$

with *Ii* defined as the scalar moment of inertia of the *i*th link about an axis passing through its center of mass, in the direction normal to the plane of motion, while **0** is the 2-dimensional zero vector and **1** is the 2×2 identity matrix.

Furthermore, the Newton-Euler equations of the *ith* link take on the forms

$$n_i = I_i\dot{\omega}_i \qquad \mathbf{f}_\mathbf{i} = m_\mathrm{i}\ddot{\mathbf{c}}_\mathbf{i}$$

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These equations can now be cast in the form

$$\mathbf{M}_i \dot{\mathbf{t}}_i = \mathbf{w}_i^W + \mathbf{w}_i^C$$

where we have decomposed the total wrench acting on the ith link into its working component $\mathbf{w}_i^W$, supplied by the environment and accounting for motor and joint dissipative torques, and $\mathbf{w}_i^C$, the nonworking constraint wrench, supplied by the neighboring links via the coupling joints. The latter, it is recalled, develop no power, their sole role being to keep the links together.

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An essential difference from the general 6-D counterpart of the foregoing equation, is the lack of a quadratic term in ωi and consequently, the lack of a $\mathbf{W}_i\mathbf{M}_i\mathbf{t}_i$ term.

Upon assembling the foregoing 3n equations of motion, we obtain a system of 3n uncoupled equations in the form:

$$\mathbf{M}\dot{\mathbf{t}} = \mathbf{w}^{W} + \mathbf{w}^{C}$$

Now, the wrench $\mathbf{w}^{W}$ accounts for active forces and moments exerted on the manipulator, and so we can decompose this wrench into an actuator supplied wrench $\mathbf{w}^{A}$,and a gravity wrench $\mathbf{w}^{G}$

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In the next step of the formulation, we set up the kinematic constraints in linear homogeneous form, as in equation $\mathbf{Kt=0}$, with the difference that now, in the presence of n kinematic pairs of the revolute or the prismatic type, $\mathbf{K}$ is a $3n\times3n$ matrix.

Moreover, we set up the twist-shape relations in the form of $\dot{\mathbf{t}} = \mathbf{T}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}$ except that now, $\mathbf{T}$ is a 3n x n matrix.

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The derivation of the Euler-Lagrange equations for planar motion using the natural orthogonal complement, then, parallels that of general 3-dimensional motion, the model sought taking the form

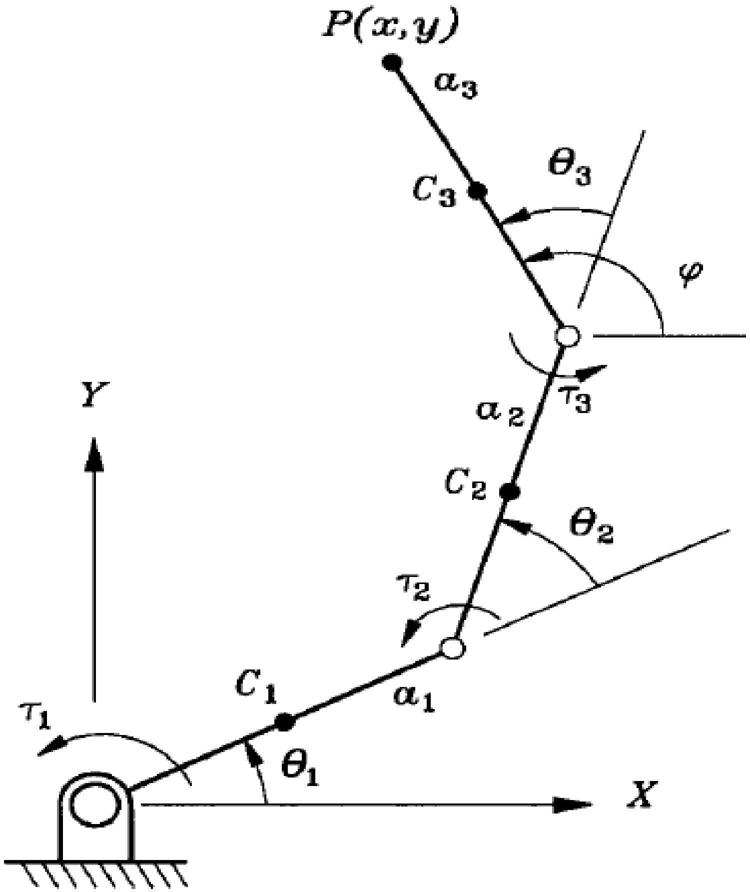
$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} + \boldsymbol{\gamma} + \boldsymbol{\delta}$$

with the definitions:

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{T}^{T}\mathbf{M}\mathbf{T} \qquad \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \mathbf{T}^{T}\mathbf{M}\dot{\mathbf{T}}$$

$$\boldsymbol{\tau} = \mathbf{T}^{T}\mathbf{w}^{A} \quad \boldsymbol{\gamma} = \mathbf{T}^{T}\mathbf{w}^{G} \quad \boldsymbol{\delta} = \mathbf{T}^{T}\mathbf{w}^{D}$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR



Derive the model of the manipulator of figure using the natural orthogonal complement.

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

We start by deriving all kinematics-related variables, and thus,

$$\omega_1 = \dot{\theta}_1$$
$$\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$$
$$\omega_3 = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$
$$\mathbf{t}_1 = \dot{\theta}_1 \mathbf{t}_{11}$$
$$\mathbf{t}_2 = \dot{\theta}_1 \mathbf{t}_{21} + \dot{\theta}_2 \mathbf{t}_{22}$$
$$\mathbf{t}_3 = \dot{\theta}_1 \mathbf{t}_{31} + \dot{\theta}_2 \mathbf{t}_{32} + \dot{\theta}_3 \mathbf{t}_{33}$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

Where:

$$\mathbf{t}_{11} = \begin{bmatrix} 1 \\ \mathbf{Er}_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}\boldsymbol{\rho}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ (1/2)\mathbf{Ea}_1 \end{bmatrix}$$

$$\mathbf{t}_{21} = \begin{bmatrix} 1 \\ \mathbf{Er}_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_1 + \boldsymbol{\rho}_2) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_1 + (1/2)\mathbf{a}_2) \end{bmatrix}$$

$$\mathbf{t}_{22} = \begin{bmatrix} 1 \\ \mathbf{Er}_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}\boldsymbol{\rho}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ (1/2)\mathbf{Ea}_2 \end{bmatrix}$$

$$\mathbf{t}_{31} = \begin{bmatrix} 1 \\ \mathbf{Er}_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_1 + \mathbf{a}_2 + \boldsymbol{\rho}_3) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_1 + \mathbf{a}_2 + (1/2)\mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{t}_{32} = \begin{bmatrix} 1 \\ \mathbf{Er}_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_2 + \boldsymbol{\rho}_3) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{E}(\mathbf{a}_2 + (1/2)\mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{t}_{33} = \begin{bmatrix} 1 \\ \mathbf{Er}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{E}\boldsymbol{\rho}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ (1/2)\mathbf{Ea}_3 \end{bmatrix}$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

Hence, the 9 x 3 twist-shaping matrix **T** becomes:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ (1/2)\mathbf{E}\mathbf{a}_1 & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 0 \\ \mathbf{E}(\mathbf{a}_1 + (1/2)\mathbf{a}_2) & (1/2)\mathbf{E}\mathbf{a}_2 & \mathbf{0} \\ 1 & 1 & 0 \\ \mathbf{E}(\mathbf{a}_1 + \mathbf{a}_2 + (1/2)\mathbf{a}_3 & \mathbf{E}(\mathbf{a}_2 + (1/2)\mathbf{a}_3) & (1/2)\mathbf{E}\mathbf{a}_3 \end{bmatrix}$$

The 9×9 matrix of inertia dyads of this manipulator now takes the form

$$\mathbf{M} = \mathrm{diag}\,(\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3)$$

with each 3 × 3 $\mathbf{M}_i$ matrix defined as $\mathbf{M}_i = \begin{bmatrix} I_i & \mathbf{0}^T \\ \mathbf{0} & m_i\mathbf{1} \end{bmatrix}$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

Now, the 3 x 3 generalized inertia matrix is readily derived as : $\mathbf{I} = \mathbf{TMT}^T$

whose entries are given below:

$$I_{11} = \mathbf{t}_{11}^T \mathbf{M}_1 \mathbf{t}_{11} + \mathbf{t}_{21}^T \mathbf{M}_2 \mathbf{t}_{21} + \mathbf{t}_{31}^T \mathbf{M}_3 \mathbf{t}_{31}$$
$$I_{12} = \mathbf{t}_{21}^T \mathbf{M}_2 \mathbf{t}_{22} + \mathbf{t}_{31}^T \mathbf{M}_3 \mathbf{t}_{32} = I_{21}$$
$$I_{13} = \mathbf{t}_{31}^T \mathbf{M}_3 \mathbf{t}_{33} = I_{31}$$
$$I_{22} = \mathbf{t}_{22}^T \mathbf{M}_2 \mathbf{t}_{22} + \mathbf{t}_{32}^T \mathbf{M}_3 \mathbf{t}_{32}$$
$$I_{23} = \mathbf{t}_{32}^T \mathbf{M}_3 \mathbf{t}_{33} = I_{32}$$
$$I_{33} = \mathbf{t}_{33}^T \mathbf{M}_3 \mathbf{t}_{33}$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - RESULT

Upon expansion, the above entries result in exactly the same expressions as those derived in Example 6.3.1, thereby confirming the correctness of the two derivations. Furthermore, the next term in the Euler-Lagrange equations is derived below. Here, we will need $\dot{\mathbf{T}}$, which is readily derived from the above expression for $\mathbf{T}$.

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

$\dot{\mathbf{T}}$ is readily derived from the above expression for $\mathbf{T}$.

$$T = \begin{bmatrix} 0 & 0 & 0 \\ (1/2)\dot{\theta}_1\mathbf{a}_1 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 \\ \dot{\theta}_1\mathbf{a}_1 + (1/2)\,\dot{\theta}_{12}\mathbf{a}_2 & (1/2)\,\dot{\theta}_{12}\mathbf{a}_2 & \mathbf{0} \\ 0 & 0 & 0 \\ \dot{\theta}_1\mathbf{a}_1 + \dot{\theta}_{12}\mathbf{a}_2 + (1/2)\,\dot{\theta}_{123}\mathbf{a}_3 & \dot{\theta}_{12}\mathbf{a}_2 + (1/2)\,\dot{\theta}_{123}\mathbf{a}_3 & (1/2)\,\dot{\theta}_{123}\mathbf{a}_3 \end{bmatrix}$$

where: $\dot{\theta}_{12} = \dot{\theta}_1 + \dot{\theta}_2$ and $\dot{\theta}_{123} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

We now can perform the product $\mathbf{T}^T\mathbf{M}\dot{\mathbf{T}}$, whose (i, j) entry will be represented as μ_{ij}. Below we display the expressions for these entries

$$\mu_{11} = -\frac{1}{2}[m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_2 - \frac{m_3}{2}(a_1a_3s_{23} + a_2a_3s_3)\dot{\theta}_3$$

$$\mu_{12} = -\frac{1}{2}[[m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_1 + \dot{\theta}_2 - \frac{m_3}{2}(a_1a_3s_{23} + a_2a_3s_3)\dot{\theta}_3$$

$$\mu_{13} = -\frac{1}{2}m_3(a_1a_3s_{23} + a_2a_3s_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$\mu_{21} = \frac{1}{2}[m_2a_1a_2s_2 + m_3(2a_1a_2s_2 + a_1a_3s_{23})]\dot{\theta}_1 - \frac{1}{2}m_3a_2a_3s_3\dot{\theta}_3$$

$$\mu_{22} = -\frac{1}{2}m_3a_2a_3s_3\dot{\theta}_3$$

$$\mu_{23} = -\frac{1}{2}m_3a_2a_3s_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

$$\mu_{31} = \frac{1}{2}[m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_1 + a_2 a_3 s_3 \dot{\theta}_2$$

$$\mu_{32} = \frac{1}{2} m_3 a_2 a_3 s_3 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\mu_{33} = 0$$

Now, let us define: $\boldsymbol{\nu} = \mathbf{T}^T \mathrm{M} \dot{\mathbf{T}} \dot{\boldsymbol{\theta}}$

$$\nu_1 = -[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1 \dot{\theta}_2 - m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_1 \dot{\theta}_3 + \\ - \frac{1}{2}[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_2^2 - m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)\dot{\theta}_2 \dot{\theta}_3 + \\ - \frac{1}{2} m_3(a_1 a_3 s_{23} + +a_1 a_2 s_3)\dot{\theta}_3^2$$

$$\nu_2 = \frac{1}{2}[m_2 a_1 a_2 s_2 + m_3(2a_1 a_2 s_2 + a_1 a_3 s_{23})]\dot{\theta}_1 - m_3 a_2 a_3 s_3 \dot{\theta}_1 \dot{\theta}_3 - m_3 a_2 a_3 s_3 \dot{\theta}_2 \dot{\theta}_3 - \frac{1}{2} m_3 a_2 a_3 s_3 \dot{\theta}_3^2$$

$$\nu_3 = \frac{1}{2} m_3(a_1 a_3 s_{23} + a_2 a_3 s_3)]\dot{\theta}_1^2 + m_3\, a_2 s_{23} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_3 a_2 a_3 s_3 \dot{\theta}_3^2$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

The mathematical model sought, thus, takes the form:

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{\nu}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \boldsymbol{\tau} + \boldsymbol{\gamma}$$

where $\boldsymbol{\delta} = \mathbf{0}$ because we have not included dissipation. Moreover, $\boldsymbol{\gamma}$ is derived as described below.

Let $\mathbf{w}_i^G$ be the gravity wrench acting on the ith link, we have:

$$\mathbf{w}^G = \begin{bmatrix} \mathbf{w}_1^G \\ \mathbf{w}_2^G \\ \mathbf{w}_3^G \end{bmatrix} \quad \mathbf{w}_1^G = \begin{bmatrix} 0 \\ -m_1 g \mathbf{j} \end{bmatrix} \quad \mathbf{w}_2^G = \begin{bmatrix} 0 \\ -m_2 g \mathbf{j} \end{bmatrix} \quad \mathbf{w}_3^G = \begin{bmatrix} 0 \\ -m_3 g \mathbf{j} \end{bmatrix}$$

DYNAMICS OF A PLANAR 3-R MANIPULATOR - SOLUTION

Therefore, we have:

$$\boldsymbol{\gamma} = \mathbf{T}^T \mathbf{w}^G$$

$$\gamma = \frac{g}{2}\begin{bmatrix} m_1\mathbf{a}_1^T\mathbf{Ej} + m_2(2\mathbf{a}_1 + \mathbf{a}_2)^T\mathbf{Ej} + m_3[\,2(\mathbf{a}_1 + \mathbf{a}_2) + \boldsymbol{a}_3]^T\mathbf{Ej} \\ m_2\mathbf{a}_1^T\mathbf{Ej} + m_3(2\mathbf{a}_2 + \mathbf{a}_3)^T\mathbf{Ej} \\ m_3\mathbf{a}_3^T\mathbf{Ej} \end{bmatrix}$$

but:

$$\mathbf{a}_1^T\mathbf{Ej} = -\mathbf{a}_1^T\mathbf{i} = -a_1\cos\theta_1$$
$$\mathbf{a}_2^T\mathbf{Ej} = -\mathbf{a}_2^T\mathbf{i} = -a_2\cos(\theta_1 + \theta_2)$$
$$\mathbf{a}_3^T\mathbf{Ej} = -\mathbf{a}_3^T\mathbf{i} = -a_3\cos(\theta_1 + \theta_2 + \theta_3)$$

Hence:

$$\boldsymbol{\gamma} = \frac{g}{2}\begin{bmatrix} -m_1a_1c_1 - 2m_2(a_1c_1 + a_2c_{12}) - 2m_3(a_1c_1 + a_2c_{12} + a_3c_{123}) \\ -m_2a_2c_{12} - 2m_3(a_2c_{12} + a_3c_{123}) \\ -m_3a_3c_{123} \end{bmatrix}$$