DYNAMICS OF PARALLEL MANIPULATOR

The $6n$x$6n$ matrices of manipulator mass **M** and manipulator angular velocity **W** are introduced below:

$$\mathbf{M} = \mathrm{diag}\,(\mathbf{M}_1, \mathbf{M}_2, \dots \dots, \mathbf{M}_n)$$
$$\mathbf{W} = \mathrm{diag}\,(\mathbf{W}_1, \mathbf{W}_2, \dots \dots, \mathbf{W}_n)$$

From this definitions we have:

$$\boldsymbol{\mu} = \mathbf{M}\mathbf{t}$$
$$\dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t}$$

With the foregoing definitions, then, the kinetic energy of the manipulator takes on a simple form, namely,

$$T = \frac{1}{2}\mathbf{t}^T\mathbf{M}\mathbf{t} = \frac{1}{2}\mathbf{t}^T\boldsymbol{\mu}$$

Upon deriving from the expression of the kinetic energy we have:

$$\frac{\partial \boldsymbol{T}}{\partial \dot{\boldsymbol{\theta}}} = \frac{1}{2}\left(\frac{\partial \boldsymbol{t}}{\partial \dot{\boldsymbol{\theta}}}\right)^T \boldsymbol{\mu} + \frac{1}{2}\left(\frac{\partial \boldsymbol{\mu}}{\partial \dot{\boldsymbol{\theta}}}\right)^T \mathbf{t}$$

$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}}$ ➔

$$\frac{\partial \mathbf{t}}{\partial \dot{\boldsymbol{\theta}}} = \mathbf{T}$$

$$\frac{\partial \boldsymbol{\mu}}{\partial \dot{\boldsymbol{\theta}}} = M\frac{\partial \boldsymbol{t}}{\partial \dot{\boldsymbol{\theta}}} = \mathbf{MT}$$

$$\frac{\partial \boldsymbol{T}}{\partial \dot{\boldsymbol{\theta}}} = \boldsymbol{T}^T \boldsymbol{\mu}$$

$$\frac{d}{dt}\left(\frac{\partial \mathbf{T}}{\partial \dot{\boldsymbol{\theta}}}\right) = \dot{\mathbf{T}}^{T}\mu + \mathbf{T}^{T}\dot{\mu}$$

$$\frac{d}{dt}\left(\frac{\partial \mathbf{T}}{\partial \dot{\boldsymbol{\theta}}}\right) = \dot{\mathbf{T}}^{T}\mathbf{M}\mathbf{t} + \mathbf{T}^{T}\mathbf{M}\dot{\mathbf{t}} + \mathbf{T}^{T}\mathbf{W}\mathbf{M}\mathbf{t}$$

$$\frac{d}{dt}\left(\frac{\partial \mathbf{T}}{\partial \dot{\boldsymbol{\theta}}}\right) = \dot{\mathbf{T}}^{T}\mathbf{M}\mathbf{T}\dot{\boldsymbol{\theta}} + \dot{\mathbf{T}}^{T}\mathbf{M}\dot{\mathbf{T}}\dot{\boldsymbol{\theta}} + \dot{\mathbf{T}}^{T}\mathbf{M}\mathbf{T}\ddot{\boldsymbol{\theta}} + \mathbf{T}^{T}\mathbf{W}\mathbf{M}\mathbf{T}\dot{\boldsymbol{\theta}}$$

The Euler-Lagrange equations can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial T}{\partial \boldsymbol{\theta}} + \frac{\partial V}{\partial \boldsymbol{\theta}} = \phi_n$$

$$\dot{\mathbf{T}} = \frac{1}{2}\boldsymbol{\mu}^T\dot{\mathbf{t}} + \frac{1}{2}\mathbf{t}^t\dot{\boldsymbol{\mu}}$$

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}} \qquad \dot{\mathbf{t}} = \mathbf{T}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t} = \mathbf{M}\mathbf{T}\ddot{\boldsymbol{\theta}} + \mathbf{M}\dot{\mathbf{T}}\dot{\boldsymbol{\theta}} + \mathbf{W}\mathbf{M}\mathbf{T}\dot{\boldsymbol{\theta}}$$

$$\dot{\mathbf{T}} = \mathbf{t}^{\mathbf{T}}\mathbf{M}\mathbf{T}\ddot{\boldsymbol{\theta}} + \mathbf{t}^{\mathbf{T}}\mathbf{M}\dot{\mathbf{T}}\dot{\boldsymbol{\theta}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{t}^{\mathbf{T}}\mathbf{W}\mathbf{M}\mathbf{T}\dot{\boldsymbol{\theta}}$$

$$\mathbf{W}\mathbf{t} = \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_r \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_r \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1\mathbf{t}_1 \\ \mathbf{W}_2\mathbf{t}_2 \\ \vdots \\ \mathbf{W}_r\mathbf{t}_r \end{bmatrix} \Rightarrow \mathbf{W}\mathbf{t} = 0$$

After the semplification, the equation takes the form:

$$\dot{\mathbf{T}} = \mathbf{t}^{\mathrm{T}}\mathbf{M}\mathbf{T}\ddot{\boldsymbol{\theta}} + \mathbf{t}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{T}}\dot{\boldsymbol{\theta}}$$

Recalling foregoing relation, we have:

$$\frac{\partial \mathbf{T}}{\partial \boldsymbol{\theta}} = \mathbf{T}^{T}\mathbf{M}\mathbf{t} = \mathbf{T}^{T}\mathbf{M}\mathbf{T}\dot{\boldsymbol{\theta}}$$

Upon substituting , we obtain:

$$\dot{\mathbf{T}} = \left(\frac{\partial \mathbf{T}}{\partial \boldsymbol{\theta}}\right)^{\boldsymbol{T}} \dot{\theta} + \left(\frac{\partial \mathbf{T}}{\partial \dot{\boldsymbol{\theta}}}\right)^{\boldsymbol{T}} \ddot{\theta}$$

$$\frac{\partial \prod}{\partial \dot{\boldsymbol{\theta}}} = \frac{\partial}{\partial \dot{\boldsymbol{\theta}}}(\mathbf{t}^T \mathbf{w}^A) = \left(\frac{\partial \mathbf{t}}{\partial \dot{\boldsymbol{\theta}}}\right)^T \mathbf{w}^A$$

$$\frac{\partial \prod}{\partial \dot{\boldsymbol{\theta}}} = \mathbf{T}^T \mathbf{w}^A$$

$$\frac{\partial \boldsymbol{\Delta}}{\partial \dot{\boldsymbol{\theta}}} = \mathbf{T}^T \mathbf{w}^D \qquad \mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}(\theta \quad \dot{\theta})\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} - \boldsymbol{\delta}$$

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{T}^T \mathbf{M} \mathbf{T}$$

Substituting these relations in the equation we have:

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{T}^T \mathbf{M}\, \dot{\mathbf{T}} \dot{\boldsymbol{\theta}} + \mathbf{T}^T \mathbf{W} \mathbf{M} \mathbf{T}\, \dot{\boldsymbol{\theta}} = \mathbf{T}^T (\mathbf{w}^A - \mathbf{w}^D)$$

$$\mathbf{I}\ddot{\boldsymbol{\theta}} = -\mathbf{T}^T \mathbf{M}\, \dot{\mathbf{T}} \dot{\boldsymbol{\theta}} - \mathbf{T}^T \mathbf{W} \mathbf{M} \mathbf{T}\, \dot{\boldsymbol{\theta}} + \mathbf{T}^T (\mathbf{w}^A - \mathbf{w}^D)$$

The equation can be expressed in the form below:

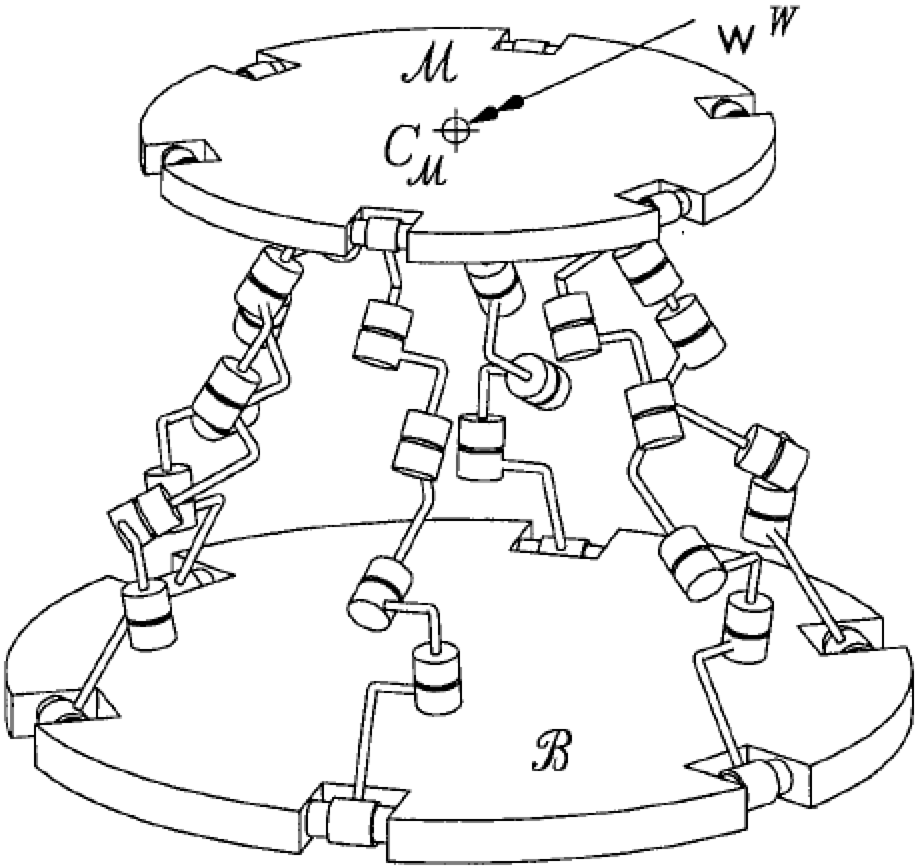
$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \boldsymbol{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} - \boldsymbol{\delta}$$

INTRODUCTION

We illustrate the modeling techniques of mechanical systems with kinematic loops via a class of systems known as parallel manipulators. While parallel manipulators can take on a large variety of forms, we focus here on those termed platform manipulators, with an architecture similar to that of flight simulators.

In platform manipulators we can distinguish two special links, namely, the base $\mathcal{B}$ and the moving platform $\mathcal{M}$. Moreover, these two links are coupled via six legs, with each leg constituting a six-axis kinematic chain of the serial type, as shown in figure whereby a wrench $\mathbf{w}^W$, represented by a double-headed arrow, acts on $\mathcal{M}$ and is applied at C_M the mass center of $\mathcal{M}$.

PARALLEL MANIPULATOR



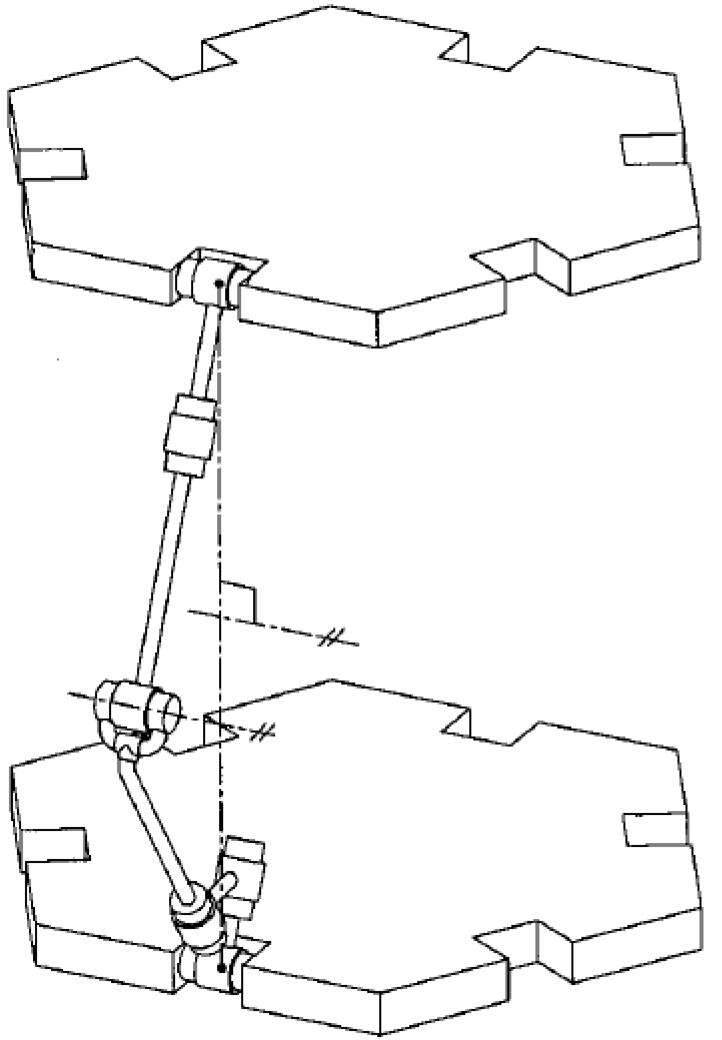
PARALLEL MANIPULATOR

However, the modeling discussed below is not restricted to this particular geometry. As a matter of fact, these axes need not even be coplanar. On the other hand, the architecture of the above figure is very general, for it includes more specific types of platform manipulators, such as flight simulators. In these, the first three revolute axes stemming from the base platform have intersecting axes, thereby giving rise to a spherical kinematic pair, while the upper two axes intersect at right angles, thus constituting a universal joint. Moreover, the intermediate joint in flight simulators is not a revolute, but rather a prismatic

PARALLEL MANIPULATOR

A leg kinematically equivalent to that of flight simulators can be obtained from that of the manipulator if the intermediate revolute has an axis perpendicular to the line connecting the centers of the spherical and the universal joints of the corresponding leg. In flight simulators, the pose of the moving platform is controlled by hydraulic actuators that vary the distance between these two centers. In the revolute-coupled equivalent leg, the length of the same line is controlled by the rotation of the intermediate revolute.

PARALLEL MANIPULATOR



The graph of the system depicted in the above figure

EULER’S FORMULA FOR GRAPHS

The number ι of independent loops of a system with many kinematic loops is given by:

$$\iota = j - l + 1$$

where j is the number of revolute and prismatic joints and l is the number of links.

Thus, if we apply Euler's formula to the system we conclude that its kinematic chain contains 5 independent loops. Hence, while the chain apparently contains 6 distinct loops, only 5 of these are independent.

EULER'S FORMULA FOR GRAPHS

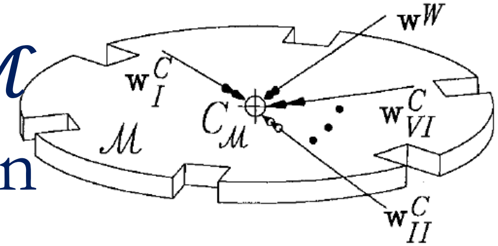
Moreover, the degree of freedom of the manipulator is six. Indeed, the total number of links of the manipulator is l=6x5+2 = 32. Of these, one is fixed, and hence, we have 31 moving links, each with six degrees of freedom prior to coupling. Thus, we have a total of 31x6 = 186 degrees of freedom at our disposal.

Upon coupling, each revolute removes 5 degrees of freedom, and hence, the 36 kinematic pairs remove 180 degrees of freedom, the manipulator thus being left with 6 degrees of freedom. the legs.

We assume that each leg is a six-axis open kinematic chain with either revolute or prismatic pairs, only one of which is actuated, and we thus have as many actuated joints as degrees of freedom. We label the legs with Roman numerals *I*, *II*, …., *VI* and denote the mass center of the mobile platform $\mathcal{M}$ by $C_{\mathcal{M}}$ with the twist of $\mathcal{M}$ denoted by $\mathbf{t}_{\mathcal{M}}$ and defined at the mass center. That is, if $\mathbf{c}_{\mathcal{M}}$ denotes the position vector of $C_{\mathcal{M}}$ in an inertial frame and $\dot{\mathbf{c}}_{\mathcal{M}}$ its velocity, while $\boldsymbol{\omega}_{\mathcal{M}}$ is the angular velocity of $\mathcal{M}$, then:

$$\mathbf{t}_{\mathcal{M}} = \begin{bmatrix} \boldsymbol{\omega}_{\mathcal{M}} \\ \dot{\mathbf{c}}_{\mathcal{M}} \end{bmatrix}$$

DYNAMICS OF PARALLEL MANIPULATORS

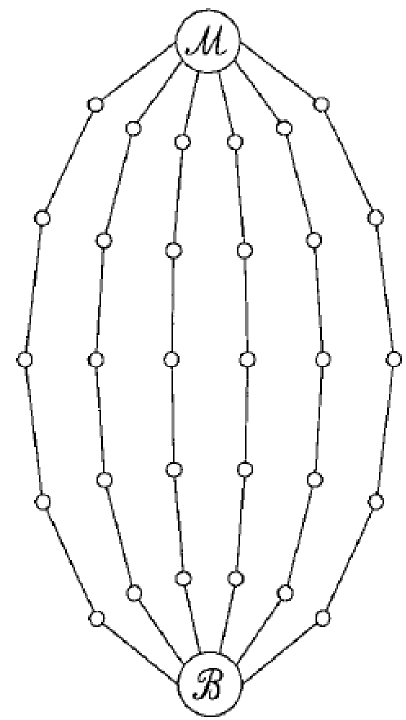


Next, the Newton-Euler equations of $\mathcal{M}$ are derived from the free-body diagram . In this figure, the legs have been replaced by the constraint wrenches $\{\mathbf{w}_J{}^C\}_I{}^{VI}$ acting at point $C_{\mathcal{M}}$.

The governing equation takes the form: (*****)

$$\mathbf{M}_{\mathcal{M}}\dot{\mathbf{t}}_{\mathcal{M}} = -\mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{t}_{\mathcal{M}} + \mathbf{w}^W + \sum_{j=I}^{VI}\mathbf{w}_J^C$$

with $\mathbf{w}^W$ denoting the external wrench acting on $\mathcal{M}$.



DYNAMICS OF PARALLEL MANIPULATORS

Let us denote by q_J the variable of the actuated joint of the Jth leg, all variables of the six actuated joints being grouped in the 6-dimensional array $\mathbf{q}$, i.e.,

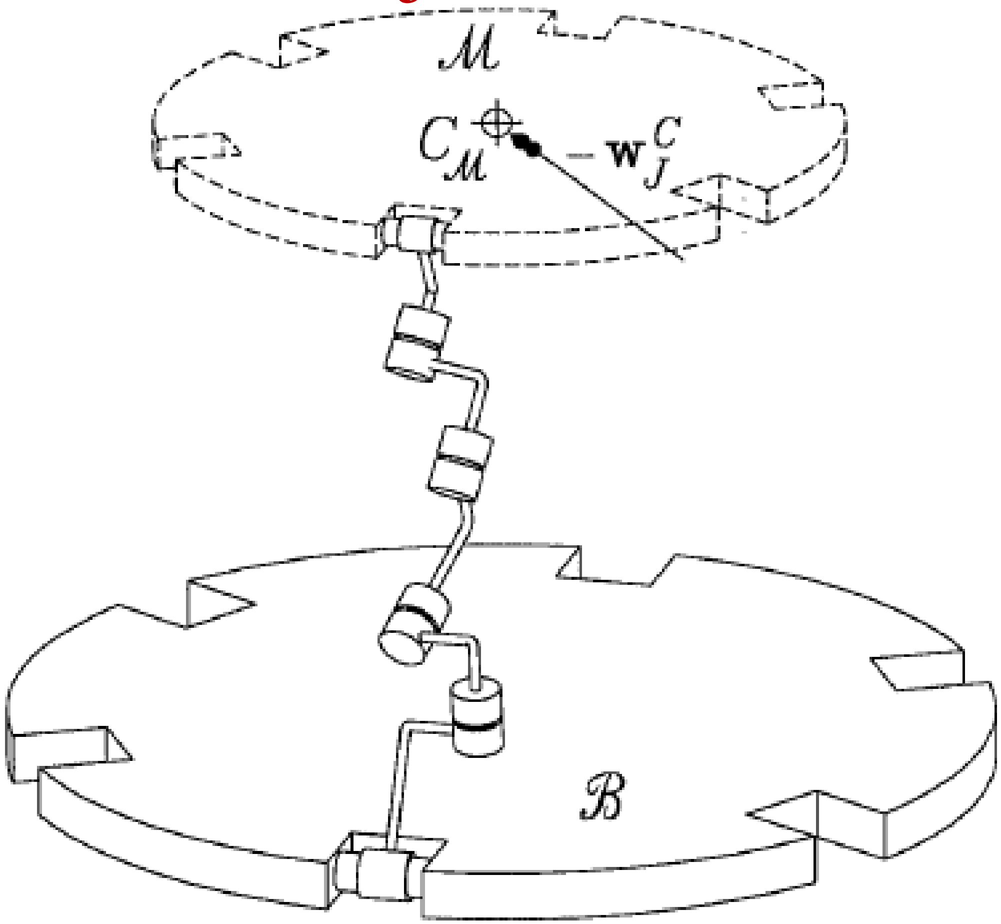
$$\mathbf{q} \equiv [q_I \quad q_{II} \quad \cdots \quad q_{VI}]^T$$

We derive a relation between the twist $\mathbf{t}_{\mathcal{M}}$ and the active joint rates, $\dot{q}_J$ for $J = I, II, ..., IV$.

To this end, we resort to the next figure, depicting the Jth leg as a serial-type, six-axis manipulator, whose twist-shape relations are readily expressed as:

$$\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_{\mathcal{M}} \qquad \text{for } J = I, II, ..., VI$$

THE SERIAL MANIPULATOR OF THE *J*TH LEG



DYNAMICS OF PARALLEL MANIPULATORS

The moving platform $\mathcal{M}$ has been replaced by the constraint wrench transmitted by the moving platform onto the end link of the Jth leg, $-\mathbf{w}_J^C$, whose sign is the opposite of that transmitted by this leg onto $\mathcal{M}$ by virtue of Newton's third law. The dynamics model of the manipulator then takes the form:

$$\mathbf{I}_J\ddot{\boldsymbol{\theta}}_J + \mathbf{C}_J(\boldsymbol{\theta}_J, \dot{\boldsymbol{\theta}}_J)\dot{\boldsymbol{\theta}}_J = \boldsymbol{\tau}_J - \mathbf{J}_J^T\mathbf{w}_J^C$$

Where where $\mathbf{I}_J$ is the 6x6 inertia matrix of the manipulator, while $\mathbf{C}_J$ is the matrix coefficient of the inertia terms that are quadratic in the joint rates.

DYNAMICS OF PARALLEL MANIPULATORS

Moreover, $\boldsymbol{\theta}_J$ and $\boldsymbol{\tau}_J$ denote the 6-dimensional vectors of joint variables and joint torques, namely,

$$\boldsymbol{\theta}_J \equiv \begin{bmatrix} \theta_{J1} \\ \theta_{J2} \\ \vdots \\ \theta_{J6} \end{bmatrix} \qquad \boldsymbol{\tau}_J \equiv \begin{bmatrix} 0 \\ \vdots \\ \tau_{Jk} \\ \vdots \\ 0 \end{bmatrix}$$

with subscript Jk denoting in turn the only actuated joint of the Jth leg, namely, the kth joint of the leg. If we now introduce $\mathbf{e}_{jk}$, defined as a unit vector all of whose entries are zero except for the kth entry, which is unity, then we can write :

$$\boldsymbol{\tau}_J = f_J \mathbf{e}_{Jk}$$

DYNAMICS OF PARALLEL MANIPULATORS

If the actuated joint is prismatic, as is the case in flight simulators, then f_J is a force; if this joint is a revolute, then f_J is a torque.

Now, since the dimension of $\mathbf{q}$ coincides with the degree of freedom of the manipulator, it is possible to find, within the framework of the natural orthogonal complement, a 6x6 matrix $\mathbf{L}_J$ mapping the vector of actuated joint rates $\dot{\mathbf{q}}$ into the vector of Jth-leg joint-rates, namely,

$$\dot{\boldsymbol{\theta}}_J = \mathbf{L}_J \dot{\mathbf{q}} \qquad J = I, II, \ldots, VI$$

DYNAMICS OF PARALLEL MANIPULATORS

Moreover, if the manipulator is not at a singular configuration, then we can solve for $\mathbf{w}_J^C$:

$$\mathbf{w}_J^C = \mathbf{J}_J^{-T}(\boldsymbol{\tau}_J - \mathbf{I}_J\ddot{\boldsymbol{\theta}}_J - \mathbf{C}_J\dot{\boldsymbol{\theta}}_J)$$

in which the superscript -T stands for the transpose of the inverse.

Further, we substitute $\mathbf{w}_J^C$ in the governing equation thereby obtaining the Newton-Euler equations of the moving platform free of constraint wrenches.

DYNAMICS OF PARALLEL MANIPULATORS

Additionally, the equations (*) thus resulting now contain inertia terms and joint torques pertaining to the *J*th leg, namely,

$$\mathbf{M}_{\mathcal{M}}\dot{\mathbf{t}}_{\mathcal{M}} = -\mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{t}_{\mathcal{M}} + \mathbf{w}^{W} + \sum_{J=I}^{VI}\mathbf{J}^{-T}(\boldsymbol{\tau}_J - \mathbf{I}_J\ddot{\boldsymbol{\theta}}_J - \mathbf{C}_J\dot{\boldsymbol{\theta}}_J)$$

Still within the framework of the natural orthogonal complement, we set up the relation between the twist $\mathbf{t}_M$ and the vector of actuated joint rates $\dot{\mathbf{q}}$ as:

$$\mathbf{t}_{\mathcal{M}} = \mathbf{T}\dot{\mathbf{q}}$$

DYNAMICS OF PARALLEL MANIPULATORS

Upon differentiation with respect to time, yields

$$\dot{\mathbf{t}}_{\mathcal{M}} = \mathbf{T}\ddot{\mathbf{q}} + \dot{\mathbf{T}}\dot{\mathbf{q}}$$

In the next step, we substitute $\mathbf{t}_{\mathcal{M}}$ its time-derivative into the equation (**)we obtain:

$$\mathbf{M}_{\mathcal{M}}\left(\mathbf{T}\ddot{\mathbf{q}} + \dot{\mathbf{T}}\dot{\mathbf{q}}\right) + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}\dot{\mathbf{q}} + \sum_{J=I}^{VI} \mathbf{J}^{-T}(\mathbf{I}_J\ddot{\boldsymbol{\theta}}_J - \mathbf{C}_J\dot{\boldsymbol{\theta}}_J) = \mathbf{w}^W + \sum_{J=I}^{VI} \mathbf{J}^{-T}\,\boldsymbol{\tau}_J$$

Further, we recall relation $\dot{\boldsymbol{\theta}}_J = \mathbf{L}_J\dot{\mathbf{q}}$ which upon differentiation with respect to time, yields:

$$\ddot{\boldsymbol{\theta}}_J = \mathbf{L}_J\ddot{\mathbf{q}} + \dot{\mathbf{L}}_J\dot{\mathbf{q}}$$

DYNAMICS OF PARALLEL MANIPULATORS

These relations are substituted in the equation(*******) thereby obtaining the model sought in terms only of actuated joint variables. After simplification, this model takes the form

$$\mathbf{M}_{\mathcal{M}}\mathbf{T}\,\ddot{\mathbf{q}} + \mathbf{M}_{\mathcal{M}}\dot{\mathbf{T}}\mathbf{q} + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}\dot{\mathbf{q}} + \sum_{J=I}^{VI}\mathbf{J}^{-T}(\mathbf{I}_J\mathbf{L}_J\ddot{\mathbf{q}} + \mathbf{I}_J\dot{\mathbf{L}}_J\dot{\mathbf{q}} - \mathbf{C}_J\,\mathbf{L}_J\dot{\mathbf{q}})$$

$$= \mathbf{w}^W + \sum_{J=I}^{VI}\mathbf{J}^{-T}\,\boldsymbol{\tau}_J$$

DYNAMICS OF PARALLEL MANIPULATORS

Our final step in this formulation consists in deriving a reduced 6x6 model in terms only of actuated joint variables. Prior to this step, we note that:

$\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_{\mathcal{M}}$; $\quad \dot{\boldsymbol{\theta}}_J = \mathbf{L}_J \dot{\mathbf{q}}$; $\quad \mathbf{t}_{\mathcal{M}} = \mathbf{T}\dot{\mathbf{q}} \Rightarrow \mathbf{L}_J = \mathbf{J}_J^{-1}\mathbf{T}$

Upon substitution of the above relation into the equation (****) and multiplication of both sides of equation by $\mathbf{T}^T$ from the left, we obtain the desired model in the form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}^W + \sum_{J=I}^{VI} \mathbf{L}_J \boldsymbol{\tau}_J$$

DYNAMICS OF PARALLEL MANIPULATORS

With the 6x6 matrices $\mathbf{M}(\mathbf{q}), \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$,and vector $\boldsymbol{\tau}^W$ defined as:

$$\mathbf{M}(\mathbf{q}) \equiv \mathbf{T}^T \mathbf{M}_{\mathcal{M}} \mathbf{T} + \sum_{J=I}^{VI} \mathbf{L}_J^T \mathbf{I}_J \mathbf{L}_J$$

$$\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T}^T \left(\mathbf{M}_{\mathcal{M}} \dot{\mathbf{T}} + \mathbf{W}_{\mathcal{M}} \mathbf{M}_{\mathcal{M}} \mathbf{T}_{\mathcal{M}} \right) + \sum_{J=I}^{VI} \mathbf{L}_J (\mathbf{I}_J \dot{\mathbf{L}}_J + \mathbf{C}_J \mathbf{L}_J)$$

$$\boldsymbol{\tau}^W \equiv \mathbf{T}^T \mathbf{W}^W$$

DYNAMICS OF PARALLEL MANIPULATORS

Alternatively, the foregoing variables can be expressed in a more compact form that will shed more light on the above model. To do this, we define the 36x36 matrices **I** and **C** as well as the 6x36 matrix **L**, the 6x6 matrix **Λ**, and the 6-dimensional vector $\boldsymbol{\phi}$ as:

$$\mathbf{I} = \text{diag}\,(\mathbf{I}_I, \mathbf{I}_{II}, \ldots, \mathbf{I}_{VI})$$

$$\mathbf{C} = \text{diag}\,(\mathbf{C}_I, \mathbf{C}_{II}, \ldots, \mathbf{C}_{VI})$$

$$\mathbf{L} = [\mathbf{L}_I \quad \mathbf{L}_{II} \quad \cdots \quad \mathbf{L}_{VI}]$$

$$\boldsymbol{\Lambda} = [\mathbf{L}_I\mathbf{e}_{Ik} \quad \mathbf{L}_{II}\mathbf{e}_{IIk} \quad \cdots \quad \mathbf{L}_{VI}\mathbf{e}_{VIk}]$$

$$\boldsymbol{\phi} = [f_I \quad f_{II} \quad \cdots \quad f_{VI}]^T$$

DYNAMICS OF PARALLEL MANIPULATORS

and hence:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}^W + \boldsymbol{\Lambda}\boldsymbol{\phi}$$

Thus, for inverse dynamics, we want to determine $\boldsymbol{\phi}$ for a motion given by $\mathbf{q}$ and $\dot{\mathbf{q}}$, which can be done from the above equation, namely,

$$\boldsymbol{\phi} = \boldsymbol{\Lambda}^{-1}[\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - \boldsymbol{\tau}^W]$$

Notice, however, that the foregoing solution is not recursive, and since it requires linear-equation solving, it is of order n^3, which thus yields a rather high numerical complexity. It should be possible to produce a recursive algorithm for the computation of $\boldsymbol{\phi}$.

DYNAMICS OF PARALLEL MANIPULATORS

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}^W + \boldsymbol{\Lambda}\boldsymbol{\phi}$$

For purposes of direct dynamics, on the other hand, we want to solve for $\ddot{\mathbf{q}}$. This can be readily done if we define the state-variable model thus taking on the form:

$$\dot{\mathbf{q}} = \mathbf{r}$$

$$\dot{\mathbf{r}} = \mathbf{M}^{-1}[-\mathbf{N}(\mathbf{q},\mathbf{r})\mathbf{r} + \boldsymbol{\tau}^W + \boldsymbol{\Lambda}\boldsymbol{\phi}]$$

EXAMPLE

Derive matrix $\mathbf{L}_J$ of equation for a manipulator having six identical legs and the actuators being placed at the fourth joint.

SOLUTION:We attach coordinate frames to the links of the serial chain of the Jth leg following the DH notation, while noting that the first three joints intersect at a common point, and hence, $r_1 = r_2 = r_3$. According to this notation, we recall, vector $\mathbf{r}_i$ is directed from the origin O_i of the $i$ih frame to the operation point of the manipulator, which in this case, is $C_{\mathcal{M}}$.

EXAMPLE

The Jacobian matrix of the *J*th leg then takes the form

$$\mathbf{J}_J = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_1 & \mathbf{e}_3 \times \mathbf{r}_1 & \mathbf{e}_4 \times \mathbf{r}_4 & \mathbf{e}_5 \times \mathbf{r}_5 & \mathbf{e}_6 \times \mathbf{r}_6 \end{bmatrix}$$

The subscript **J** of the array in the right-hand side reminding us that the vectors inside it pertain to the *J*th leg. Thus, matrix $\mathbf{J}_J$ maps the joint-rate vector of the *J*th leg $\dot{\boldsymbol{\theta}}_J$ into the twist $\mathbf{t}_{\mathcal{M}}$ of the platform, i.e.

$$\mathbf{J}_J \dot{\boldsymbol{\theta}}_J = \mathbf{t}_{\mathcal{M}}$$

Clearly, the joint-rate vector of the *J*th leg is defined as:

$$\dot{\boldsymbol{\theta}}_J \equiv \begin{bmatrix} \dot{\theta}_{J1} & \dot{\theta}_{J2} & \dot{\theta}_{J3} & \dot{\theta}_{J4} & \dot{\theta}_{J5} & \dot{\theta}_{J6} \end{bmatrix}^T$$

EXAMPLE

Now, note that except for $\dot{\theta}_{J4}$, all joint-rates of this leg are passive and thus need not appear in the mathematical model of the whole manipulator. Hence, we should aim at eliminating all joint-rates from the above twist-rate relation, except for the one associated with the active joint. We can achieve this if we realize that:

$$\mathbf{r}_{J1} \times \mathbf{e}_{Ji} + \mathbf{e}_{ji} \times \mathbf{r}_{J1} = \mathbf{0} \qquad i=1,2,3$$

Further, we define a 3x6 matrix $\mathbf{A}_J$ as:

$$\mathbf{A}_J = [\mathbf{R}_{J1} \quad \mathbf{1}]$$

with $\mathbf{R}_{J1}$ defined, in turn, as the cross-product matrix of $\mathbf{r}_{J1}$.

EXAMPLE

Now, upon multiplication of $\mathbf{J}_J$ by $\mathbf{A}_J$ from the left, we obtain a 3 x 6 matrix whose first three columns vanish, namely,

$$\mathbf{A}_J\mathbf{J}_J = [0 \quad 0 \quad 0 \quad \mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_4 \quad \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_5 \quad \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_5]$$

If we multiply both sides of the above twist-shape equation by $\mathbf{A}_J$ from the left, we will obtain a new twist-shape equation that is free of the first three joint rates. Moreover, this equation is 3-dimensional, i.e.,

$$\left[\mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_4 + \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_5 + \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\boldsymbol{\theta}}_5\right]$$
$$= \boldsymbol{\omega}_{\mathcal{M}} \times \mathbf{r}_{J1} + \dot{\mathbf{c}}_{\mathcal{M}}$$

Where the subscript J attached to the brackets enclosing the whole left-hand side again reminds us that all quantities therein are to be understood as pertaining to the *J*th leg.

EXAMPLE

Furthermore, only $\dot{\theta}_{J4}$ is associated with an active joint and denoted, henceforth, by $\dot{q}_J$, i.e.,

$$q_J = \theta_{J4}$$

We have now to eliminate both $\dot{\theta}_{J5}$ and $\dot{\theta}_{J6}$ from the foregoing equation. This can be readily accomplished if we dot-multiply both sides of the same equation by vector $\mathbf{u}_J$ defined as the cross product of the vector coefficients of the two passive joint rates, i.e.,

$$\mathbf{u}_J \equiv [\mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)]_J \times [\mathbf{e}_6 \times (\mathbf{r}_5 - \mathbf{r}_1)]_J$$

EXAMPLE

We thus obtain a third twist-shape relation that is scalar and free of passive joint rates, namely

$$\mathbf{u}_j \cdot [\mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\dot{\theta}_4]_J = \mathbf{u}_J(-\boldsymbol{\omega}_M \times \mathbf{r}_{J1} + \dot{\mathbf{c}}_M$$

The above equation is clearly of the form

$$\zeta_j \dot{q}_J = \mathbf{y}_J^T \mathbf{t}_{\mathcal{M}} \qquad \dot{q}_J = \left(\dot{\theta}_4\right)_J \qquad J = I, II, \ldots, VI$$

With ζ_J and $\mathbf{y}_\mathrm{J}$ defined, in turn, as:

$$\zeta_J = \mathbf{u}_\mathrm{J} \cdot \mathbf{e}_{J4} \times \left(\mathbf{r}_{J4} - \mathbf{r}_{J1}\right)$$

$$\mathbf{y}_J = \begin{bmatrix} -\mathbf{r}_1 \times \mathbf{u}_J \\ \mathbf{u}_J \end{bmatrix}$$

EXAMPLE

Upon assembling the foregoing six scalar twist-shape relations, we obtain a 6-dimensional twist-shape relation between the active joint rates of the manipulator and the twist of the moving platform, namely,

$$\mathbf{Z}\dot{\mathbf{q}} = \mathbf{Y}\mathbf{t}_{\mathcal{M}}$$

with the obvious definitions for the two 6 x 6 matrices **Y** and **Z** given below:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_I^T \\ \mathbf{y}_{II}^T \\ \vdots \\ \mathbf{y}_{VI}^T \end{bmatrix} \qquad \mathbf{Z} = \mathrm{diag}\,(\zeta_I, \zeta_{II}, \dots, \zeta_{VI})$$

EXAMPLE

We now can determine matrix $\mathbf{T}$ of the procedure described above, as long as $\mathbf{Y}$ is invertible, in the form:

$$\mathbf{T} = \mathbf{Y}^{-1}\mathbf{Z}$$

whence the leg-matrix $\mathbf{L}_J$ of the same procedure is readily determined, namely.

$$\mathbf{L}_J = \mathbf{J}_J^{-1}\mathbf{T}$$