GEOMETRY OF DECOUPLED SERIAL ROBOTS
DEFINITIONS

- **Kinematic Chain:** is a set of rigid bodies (links) coupled by kinematic pairs (joints)
- **Kinematic Pair:** is the coupling of two rigid bodies so as to constrain their relative motion. They can be produce in 2 different types:
  - *Rotating Pair R (revolute)*
  - *Sliding Pair P (prismatic)*
THE DENAVIT-HARTENBERG NOTATION

Geometry of Decoupled Serial Robots
THE DENAVIT-HARTENBERG NOTATION’S RULES

1. \( Z_i \) is the axis of the \( i^{th} \) pair.

2. \( X_i \) is defined as the common perpendicular to \( Z_{i-1} \) and \( Z_i \), directed from the former to the latter.

3. The distance between \( Z_i \) and \( Z_{i+1} \) is defined as \( a_i \).

4. The \( Z_i \)-coordinate of the intersection \( O'_i \) of \( Z_i \) with \( X_{i+1} \) is denoted by \( b_i \).

5. The angle between \( Z_i \) and \( Z_{i+1} \) is defined as \( \alpha_i \) and is measured about the positive direction of \( X_{i+1} \).

6. The angle between \( X_i \) and \( X_{i+1} \) is defined as \( \theta_i \) and is measured about the positive direction of \( Z_i \).
THE DH NOTATION

The relative position and orientation between links is fully specified by the:

- **Rotation Matrix**: taking the $X_i$, $Y_i$, $Z_i$ axes into a configuration in which they are parallel pair wise to the $X_{i+1}$, $Y_{i+1}$, $Z_{i+1}$

- **Position vector** of the origin of the latter in the former.

Let: $\lambda_i = \cos\alpha_i$, $\mu_i = \sin\alpha_i$

$$[C_i]_i \equiv \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 \\ \sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\Lambda_i]_{i'} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_i & -\mu_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}$$

Geometry of Decoupled Serial Robots
\[
[Q_i]_i = [C_i]_i[A_i]_i = \begin{bmatrix}
\cos \theta_i & -\lambda_i \sin \theta_i & \mu_i \sin \theta_i \\
\sin \theta_i & \lambda_i \cos \theta_i & -\mu_i \cos \theta_i \\
0 & \mu_i & \lambda_i
\end{bmatrix}
\]
One more factoring of matrix Qi, which finds applications in manipulator kinematics, is given below:

\[ Q_i = Z_i X_i \]

\( X_i \) and \( Z_i \) defined as two pure reflections, the former about the \( Y_i Z_i \) plane, the latter about the \( X_i Y_i \) plane

\[
[X_i] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -\lambda_i & \mu_i \\
0 & \mu_i & \lambda_i
\end{bmatrix}; \quad [Z_i] = \begin{bmatrix}
\cos \theta_i & \sin \theta_i & 0 \\
\sin \theta_i & -\cos \theta_i & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\( X_i \) and \( Z_i \) are symmetric and self-inverse
In order to derive an expression for the position vector $\mathbf{a}_i$ connecting the origin $O_i$ of $\mathcal{F}_i$ with $\mathcal{F}_{i+1}$, $O_{i+1}$ reference is made to Figure showing the relative positions of the different origins and axes involved.

From this figure: $\mathbf{a}_i \equiv \overrightarrow{O_i O_{i+1}} = \overrightarrow{O_i O_{i'}} + \overrightarrow{O_{i'} O_{i+1}}$
Where:

\[
\begin{bmatrix}
O_iO_{i+1}
\end{bmatrix}_i = \begin{bmatrix}
0 \\
0 \\
b_i
\end{bmatrix}
\]

\[
\begin{bmatrix}
O_iO_{i+1}
\end{bmatrix}_{i+1} = \begin{bmatrix}
a_i \\
0 \\
0
\end{bmatrix}
\]

The two foregoing vectors should be expressed in the same coordinate frame.

\[
\begin{bmatrix}
O_iO_{i+1}
\end{bmatrix}_i = [Q_i]_i \times \begin{bmatrix}
O_iO_{i+1}
\end{bmatrix}_{i+1} = \begin{bmatrix}
a_i \cos \theta_i \\
a_i \sin \theta_i \\
0
\end{bmatrix}
\]

Hence:

\[
[a_i]_i = \begin{bmatrix}
a_i \cos \theta_i \\
a_i \sin \theta_i \\
b_i
\end{bmatrix} = Q_i b_i ; [b_i] = \begin{bmatrix}
a_i \\
b_i \mu_i \\
b_i \lambda_i
\end{bmatrix}
\]
The kinematics of serial manipulators is about studying the:

- **Geometry**: finding relations between *joint variables* and *Cartesian variables*.
- **Relation between joint rates** and the twist of the EE.
- **the relations between joint accelerations** with the time-rate of change of the twist of the EE also pertain to robot kinematics.
We distinguish two problems:

- **Direct Displacement Problem (DDP)** where the six joint variables of a given six-axis manipulator are assumed to be known and the problem consisting in finding the pose of the EE.

- **Inverse Displacement Problem (IDP)** where, on the contrary, the pose of the EE is given, while the six joint variables that produce this pose are to be found.
SERIAL SIX-AXIS MANIPULATOR

Geometry of Decoupled Serial Robots
SERIAL SIX-AXIS MANIPULATOR

The relative position and orientation of $F_{i+1}$ with respect to $F_i$ is given by matrix $Q_i$ and vector $a_i$, respectively, which are displayed below for quick reference.

$$\begin{bmatrix} Q_i \\ a_i \end{bmatrix} = \begin{bmatrix} \cos \theta_i & -\lambda_i \sin \theta_i & \mu_i \sin \theta_i \\ \sin \theta_i & \lambda_i \cos \theta_i & -\mu_i \cos \theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} ; \begin{bmatrix} a_i \\ a_i \sin \theta_i \\ b_i \end{bmatrix}$$

$Q_i$ denotes the matrix rotating $F_i$ into an orientation coincident with that of $F_{i+1}$. $a_i$ denotes the vector joining the origin of $F_i$ with that of $F_{i+1}$, directed from the former to the latter.
SERIAL SIX-AXIS MANIPULATOR

The orientation \( \mathbf{Q} \) of the EE is obtained as a result of the six individual rotations \( \{ \mathbf{Q}_i \}_{6} \) about each revolute axis through an angle \( \theta_i \), in a sequential order, from 1 to 6.

\[
\begin{align*}
[\mathbf{Q}_6]\mathbf{a}_1 + [\mathbf{Q}_5]\mathbf{a}_2 + [\mathbf{Q}_4]\mathbf{a}_3 + [\mathbf{Q}_3]\mathbf{a}_4 + [\mathbf{Q}_2]\mathbf{a}_5 + [\mathbf{Q}_1]\mathbf{a}_6 &= [\mathbf{P}]_1 \\
\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 &= [\mathbf{P}]_1
\end{align*}
\]

Both expressed in \( \mathcal{F}_1 \).

It is hence convenient to represent the foregoing relations in each individual frame, we obtain:

\[
\mathbf{Q}_1 \mathbf{Q}_2 \mathbf{Q}_3 \mathbf{Q}_4 \mathbf{Q}_5 \mathbf{Q}_6 = \mathbf{Q}
\]
Foregoing equations can be cast in a more compact form if homogeneous transformations are now introduced. Thus, if we let $T_i = \{T_i\}_i$ be the $4 \times 4$ matrix transforming $\mathcal{F}_{i+1}$ coordinates into $\mathcal{F}_i$ -coordinates, the foregoing equations can be written in $4 \times 4$ matrix form, namely,

$$T_1 T_2 T_3 T_4 T_5 T_6 = T$$

with $T$ denoting the transformation of coordinates from the end-effector frame to the base frame. Thus, $T$ contains the pose of the end-effector.
Decoupled manipulators are those whose last three joints have intersecting axes. These joints, then, constitute the **wrist** of the manipulator, which is said to be *spherical*, because when the point of intersection \( C \) is kept fixed, all the points of the wrist move on spheres centered at \( C \).
DECOUPLED MANIPULATORS

In terms of the DH parameters of the manipulator, in a decoupled manipulator:

\[ a_4 = a_5 = b_5 = 0 \]

Thus the origins of frames 5 and 6 are coincident. For decoupled manipulators we conduct the displacement analysis by decoupling the positioning problem from the orientation problem.
Geometry of Decoupled Serial Robots

Decoupled Problem

Orientation Problem

Positioning Problem

Closed Solution of IKP
THE POSITIONING PROBLEM

Let $C$ denote the intersection of axes 4, 5, and 6, i.e., the center of the spherical wrist, and let $c$ denote the position vector of this point. The position of $C$ is independent of joint angles $\theta_4$, $\theta_5$, and $\theta_6$; hence, only the first three joints are to be considered for this analysis.
From the figure we have:

\[ a_1 + Q_1 a_2 + Q_1 Q_2 a_3 + Q_1 Q_2 Q_3 a_4 = c \]

(Expressed in \( F_1 \)-coordinates)

This equation can be readily rewritten in the form:

\[ a_2 + Q_2 a_3 + Q_2 Q_3 a_4 = Q_1^T (c - a_1) \]
THE POSITIONING PROBLEM

If we recall that $a_i = Q_i b_i$, the previous equation becomes:

$$Q_2(b_2 + Q_3 b_3 + Q_3 Q_4 b_4) = Q_1^T c - b_1$$

Since we are dealing with a decoupled manipulator, we have:

$$a_4 = Q_4 b_4 = \begin{bmatrix} 0 \\ 0 \\ b_4 \end{bmatrix} = b_4 e$$

Thus the product $Q_3 Q_4 b_4$ reduces to:

$$Q_3 Q_4 b_4 = b_4 Q_3 e = b_4 u_3$$
THE POSITIONING PROBLEM

Hence:

\[ Q_2 (b_2 + Q_3 b_3 + b_4 u_3) = Q_1^T c - b_1 \] (1)

Further, an expression for \( c \) can be derived in terms of \( p \), the position vector of the operation point of the EE, and \( Q \), namely,

\[ c = p - Q_1 Q_2 Q_3 Q_4 a_5 - Q_1 Q_2 Q_3 Q_4 Q_5 a_6 \]

Since:

\[ a_4 = a_5 = b_5 = 0 \] and \[ Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 = Q Q_6^T \]

\[ c = p - Q Q_6^T a_6 = p - Qb_6 \]
THE POSITIONING PROBLEM

The $\mathcal{F}_1$-components of $P$ and $C$ position vectors are defined as:

$$[p]_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad [c]_1 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

so $c = p - QQ_6^T a_6 = p - Qb_6$ can be expanded in the form:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - (q_{11}a_6 + q_{12}b_6\mu_6 + q_{13}b_6\lambda_6) \\ y - (q_{21}a_6 + q_{22}b_6\mu_6 + q_{23}b_6\lambda_6) \\ z - (q_{31}a_6 + q_{32}b_6\mu_6 + q_{33}b_6\lambda_6) \end{bmatrix}$$
THE POSITIONING PROBLEM

In solving the foregoing system of equations, we first note that in this equation

\[ Q_2 \left( b_2 + Q_3 b_3 + b_4 u_3 \right) = Q_1^T c - b_1 \] (1)

- the left-hand side of this equation appears multiplied by \( Q_2 \);
- \( \theta_2 \) does not appear in the right-hand side.

This implies that
- if the Euclidean norms of the two sides of that equation are equated, the resulting equation will not contain \( \theta_2 \)
- the third scalar equation of the same equation is independent of \( \theta_2 \)
THE POSITIONING PROBLEM

If we denote by \( l \) the left-hand of the equation (1) and by \( r \) its right-hand side, then we have:

\[
\begin{align*}
    l^2 & \equiv b_2^2 + b_3^2 + b_4^2 + 2b_2^T Q_3 b_3 + 2b_4 b_2^T u_3 + 2\lambda_3 b_3 \\
    r^2 & \equiv \|c\|^2 + \|b_1\|^2 - 2b_1^T Q_1^T c
\end{align*}
\]

From which it is apparent that \( l^2 \) is linear in \( x_3 \) and \( r^2 \) is linear in \( x_1 \).

Upon equating \( l^2 \) with \( r^2 \), then, an equation linear in \( x_1 \) and \( x_3 \) is readily derived, namely

\[
    Ac_1 + Bs_1 + Cc_3 + Ds_3 + E = 0
\]

Geometry of Decoupled Serial Robots
THE POSITIONING PROBLEM

\[ \begin{align*}
    Ac_1 + Bs_1 + Cc_3 + Ds_3 + E &= 0 \\
\end{align*} \]

Whose coefficients do not contain any unknown:

\[ \begin{align*}
    A &= 2a_1x_c \\
    B &= 2a_1y_c \\
    C &= 2a_2a_3 - 2b_2b_4\mu_2\mu_3 \\
    D &= 2a_3b_2\mu_2 + 2b_2b_4\mu_2\mu_3 \\
    E &= a_2^2 + a_3^2 + b_2^2 + b_3^2 + b_4^2 - a_1^2 - x_c^2 - y_c^2 \\
        &- (z_c - b_1)^2 + 2b_2b_3\lambda_3 + 2b_2b_4\lambda_2\lambda_3 + 2b_3b_4\lambda_3 
\end{align*} \]
THE POSITIONING PROBLEM

Moreover, the third scalar equation of equation (1) takes on the form:

\[ Fc_1 + Gs_1 + Hc_3 + Is_3 + J \]

Where:

\[ F = y_c \mu_1 \]
\[ G = -x_c \mu_1 \]
\[ H = -b_4 \mu_2 \mu_3 \]
\[ I = a_3 \mu_2 \]
\[ J = b_2 + b_3 \lambda_2 + b_4 \lambda_2 \lambda_3 - (z_c - b_1)\lambda_1 \]
The two above equations, can be solved for $c_3$ and $s_3$, namely:

$$c_1 = \frac{-G(Cc_3 + Ds_3 + E) + B(Hc_3 + Is_3 + J)}{\Delta_1}$$  \hspace{1cm} (2)$$

$$s_1 = \frac{F(Cc_3 + Ds_3 + E) - A(Hc_3 + Is_3 + J)}{\Delta_1}$$  \hspace{1cm} (3)$$

Where $\Delta_1$ is defined as:

$$\Delta_1 = AG - FB = -2a_1\mu_1(x_c^2 + y_c^2)$$
THE POSITIONING PROBLEM

Both sides of previous equations are squared, the squares thus obtained are then added, and the sum is equated to 1, which leads to a quadratic equation in \( x_3 \), namely:

\[
Kc_3^2 + Ls_3^2 + Mc_3s_3 + Nc_3 + Ps_3 + Q = 0
\]

whose coefficients are given below:

\[
K = (4a_1^2H^2 + \mu_1^2C^2)
\]

\[
L = (4a_1^2I^2 + \mu_1^2D^2)
\]

\[
M = 2(4a_1^2HI + \mu_1^2CD)
\]

\[
N = 2(4a_1^2HJ + \mu_1^2CE)
\]

\[
P = 2(4a_1^2IJ + \mu_1^2DE)
\]

\[
Q = (4a_1^2J^2 + \mu_1^2E^2 - 4a_1^2\mu_1^2\rho^2)
\]

Where: \( \rho^2 = x_c^2 + y_c^2 \)

Geometry of Decoupled Serial Robots
THE POSITIONING PROBLEM

Now, two well-known trigonometric identities are introduced, namely,

\[ c_3 \equiv \frac{1 - \tau_3^2}{1 + \tau_3^2} \quad \text{e} \quad s_3 \equiv \frac{2\tau_3}{1 + \tau_3^2} \quad \text{con} \quad \tau_3 \equiv \tan \left( \frac{\theta_3}{2} \right) \]

Upon substitution of the foregoing identities, a quartic equation in \( \tau_3 \) is obtained, i.e.

\[ R\tau_3^4 + S\tau_3^3 + T\tau_3^2 + U\tau_3 + V = 0 \]
THE POSITIONING PROBLEM

After some simplifications, these coefficients take on the forms:

\[
R = [4a_1^2(J - H)^2 + \mu_1^2(E - C)^2 - 4\rho^2a_1^2\mu_1^2]
\]

\[
S = 4[4a_1^2I[(J - H) + \mu_1^2D(E - C)]
\]

\[
T = 2[4a_1^2(J^2 - H^2 + 2I^2) + 2\mu_1^2(E^2 - C^2 + 2D^2) - 4\rho^2a_1^2\mu_1^2]
\]

\[
U = 4[4a_1^2I(H + J) + \mu_1^2D(C + E)]
\]

\[
V = [4a_1^2(J + H)^2 + \mu_1^2(E + C)^2 - 4\rho^2a_1^2\mu_1^2]
\]

Thus, up to four possible values of \(\theta_3\) can be obtained, namely,

\[
(\theta_3)_i = 2 \arctan[(\tau_3)_i] \quad i = 1,2,3,4
\]
THE POSITIONING PROBLEM

Once the four values of \( \theta_3 \) are available, each of these is substituted in eq. (2) and (3), which produce four different values of \( \theta_1 \).

For each value of \( \theta_1 \) and \( \theta_3 \), one value of \( \theta_2 \) can be computed from the first two scalar equations of eq.(1), which are displayed below:

\[
\begin{align*}
A_{11} \cos \theta_2 + A_{12} \sin \theta_2 &= x_c \cos \theta_1 + y_c \sin \theta_1 - a_1 \\
A_{12} \cos \theta_2 + A_{11} \sin \theta_2 &= -x_c \lambda_1 \sin \theta_1 + y_c \lambda_1 \cos \theta_1 + (z_c - b_1) \mu_1
\end{align*}
\]

Where:

\[
\begin{align*}
A_{11} &= a_2 + a_3 \cos \theta_3 + b_4 \mu_3 \sin \theta_3 \\
A_{12} &= -a_3 \lambda_2 \sin \theta_3 + b_3 \mu_2 + b_4 \lambda_2 \mu_3 \cos \theta_3 + b_4 \mu_2 \lambda_3
\end{align*}
\]
THE POSITIONING PROBLEM

If $A_{11}$ and $A_{12}$ do not vanish simultaneously, angle $\theta_2$ is readily computed as:

$$\cos \theta_2 = \frac{1}{\Delta_2} \{A_{11}(x_c \cos \theta_1 + y_c \sin \theta_1 - a_1)$$

$$- A_{12}[-x_c \lambda_1 \sin \theta_1 + y_c \lambda_1 \cos \theta_1 + (z_c - b_1) \mu_1]\}$$

$$\sin \theta_2 = \frac{1}{\Delta_2} \{A_{12}(x_c \cos \theta_1 + y_c \sin \theta_1 - a_1)$$

Where:

$$\Delta_2 = A_{11}^2 + A_{12}^2$$
THE POSITIONING PROBLEM

The four arm configurations for the positioning problem of the Puma Robot are:

(a) and (b): elbow down; (c) and (d): elbow up;
(a) and (c): shoulder fore; (b) and (d): shoulder aft.
THE ORIENTATION PROBLEM

This problem consists in determining the wrist angles that will produce a prescribed orientation of the EE. This orientation is given in terms of the rotation matrix $Q$ taking the EE from its home attitude to its current one. Alternatively, the orientation can be given by the natural invariants of the rotation matrix, vector $e$ and angle $\varphi$.

\[
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\theta_4 \\
\theta_5 \\
\theta_6 \\
\end{bmatrix}
\]
THE ORIENTATION PROBLEM

Now, since the orientation of the end-effector is given, we know the components of vector $\mathbf{e}_6$ in any coordinate frame. In particular, let

$$[\mathbf{e}_6]_4 = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad \text{and} \quad [\mathbf{e}_5]_4 = \begin{bmatrix} \mu_4 \sin \theta_4 \\ -\mu_4 \cos \theta_4 \\ \lambda_4 \end{bmatrix}$$

Vectors $\mathbf{e}_5$ and $\mathbf{e}_6$ make an angle $\alpha_5$, and hence:

$$[\mathbf{e}_6]_4^T [\mathbf{e}_5]_4 = \lambda_5$$

Geometry of Decoupled Serial Robots
THE ORIENTATION PROBLEM

Upon substitution of $[e_6]_4$ and $[e_5]_4$ in the last equation, we obtain:

$$\xi \mu_4 \sin \theta_4 - \eta \mu_4 \cos \theta_4 + \zeta \lambda_4 = \lambda_5$$

which can be readily transformed, with the aid of the tan-half-angle identities, into a quadratic equation in $\tau_4 = \tan \frac{\theta_4}{2}$

$$(\lambda_5 - \eta \mu_4 - \zeta \lambda_4) \tau_4^2 - 2\xi \mu_4 \tau_4 + (\lambda_5 + \eta \mu_4 - \zeta \lambda_4) = 0$$

$$\tau_4 = \xi \mu_4 \pm \sqrt{(\xi^2 + \eta^2) \mu_4^2 - (\lambda_5 - \zeta \lambda_4)^2}$$

$$\lambda_5 - \zeta \lambda_4 - \eta \mu_4$$

$\tau_4 \rightarrow \theta_4$

Geometry of Decoupled Serial Robots
THE ORIENTATION PROBLEM

If $Q_4 Q_5 Q_6 = R$ with $R$ defined as: $R = Q_3^T Q_2^T Q_1^T Q$

Expressions for $\theta_5$ and $\theta_6$ can be readily derived by solving first for $Q_5$, namely:

$$Q_5 = Q_4^T R Q_6^T$$

Thus, two equations for $\theta_5$ are obtained by equating the first two components of the third columns of that equation, thereby obtaining:
Thus, two equations for $\theta_5$ are obtained by equating the first two components of the third columns of that equation, thereby obtaining:

$$\mu_5 s_5 = (\mu_6 r_{12} + \lambda_6 r_{13}) c_4 + (\mu_6 r_{22} + \lambda_6 r_{23}) s_4$$

$$-\mu_5 c_5 = -\lambda_4 (\mu_6 r_{12} + \lambda_6 r_{13}) s_4 + \lambda_4 (\mu_6 r_{22} + \lambda_6 r_{23}) c_4 + \mu_4 (\mu_6 r_{32} + \lambda_6 r_{33})$$

which thus yield a unique value of $\theta_5$ for every value of $\theta_4$.  

Geometry of Decoupled Serial Robots
THE ORIENTATION PROBLEM

Finally, with $\theta_4$ and $\theta_5$ known, it is a simple matter to calculate $\theta_6$. This is done upon solving for $Q_6$ from:

$$Q_6 = Q_5^T Q_4^T R$$

Hence, $c_6$ and $s_6$ are determined as:

$$\begin{cases} 
  c_6 = \omega_1 c_5 + \omega_2 s_5 \\
  s_6 = -\omega_1 \lambda_5 s_5 + \omega_2 \lambda_5 c_5 + \omega_3 \mu_5
\end{cases}$$

Geometry of Decoupled Serial Robots
THE ORIENTATION PROBLEM

When combined with the four postures of a decoupled manipulator leading to one and the same location of its wrist center— positioning problem — a maximum of eight possible combinations of joint angles for a single pose of the end-effector of a decoupled manipulator are found