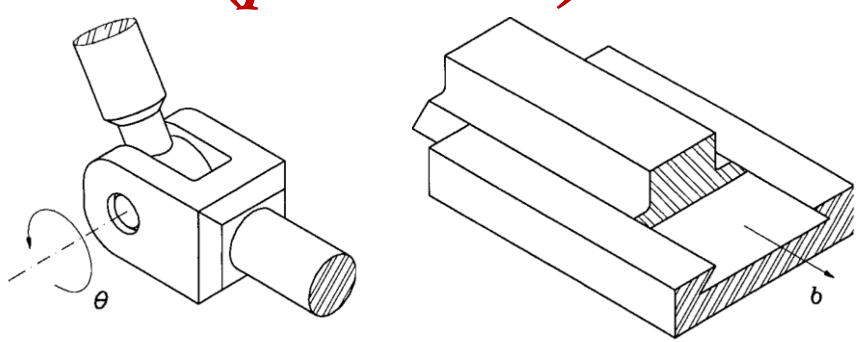
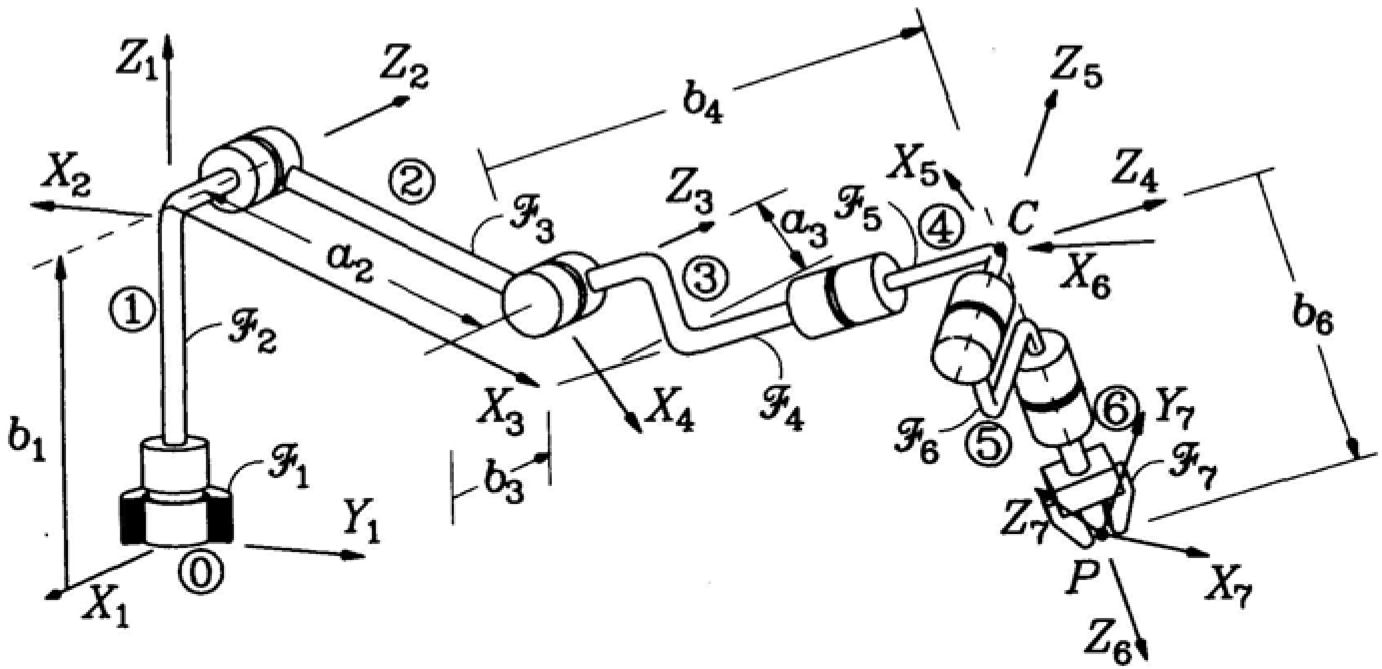
GEOMETRY OF DECOUPLED SERIAL ROBOTS

DEFINITIONS

- **Kinematic Chain:**is a set of rigid bodies(links)coupled by kinematic pairs(joints)
- **Kinematic Pair:** is the coupling of two rigid bodies so as to constrain their relative motion. They can be produce in 2 diffenrent types:
  - ***Rotating Pair R (revolute)***
  - ***Sliding Pair P (prismatic)***



THE DENAVIT-HARTENBERG NOTATION



THE DENAVIT-HARTENBERG NOTATION'S RULES

1. Z_i is the axis of the i^{th} pair.
2. X_i is defined as the common perpendicular to Z_{i-1} and Z_i, directed from the former to the latter.
3. The distance between Z_i and Z_{i+1} is defined as $\underline{a_i}$
4. The Z_i-coordinate of the intersection O'_i of Z_i with X_{i+1} is denoted by $\underline{b_i}$.
5. The angle between Z_i and Z_{i+1} is defined as $\underline{\alpha_i}$ and is measured about the positive direction of X_{i+1}.
6. The angle between X_i and X_{i+1} is defined as $\underline{\theta_i}$ and is measured about the positive direction of Z_i.

THE DH NOTATION

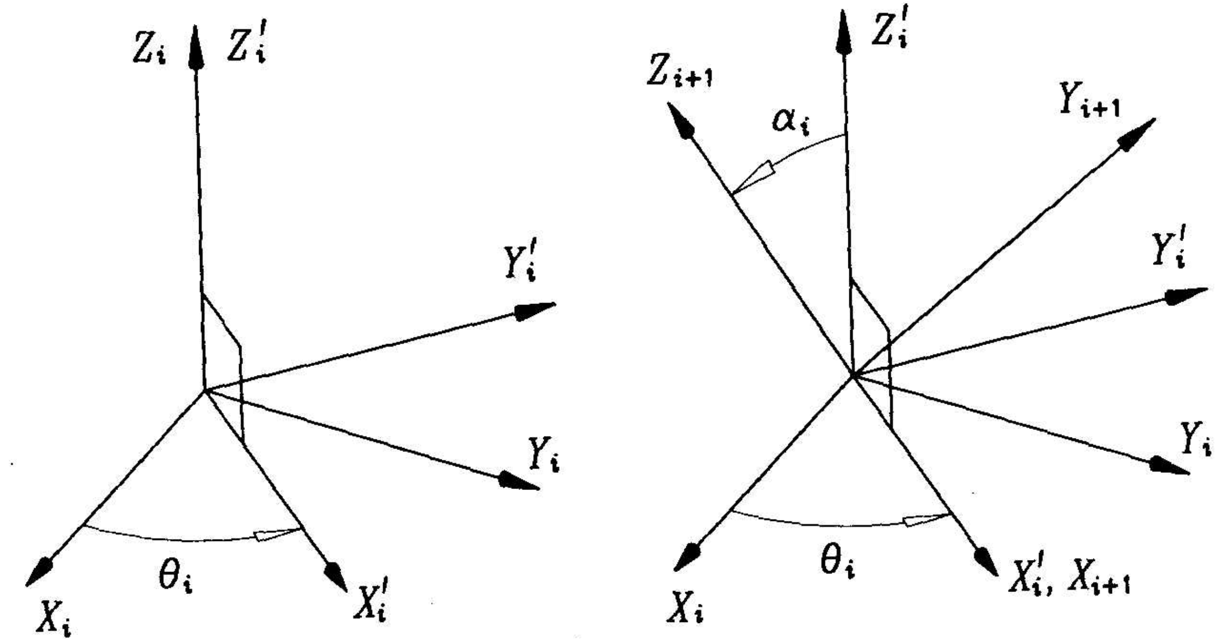
The relative position and orientation between links is fully specified by the:

- **Rotation Matrix:** taking the X_i, Y_i, Z_i axes into a configuration in which they are parallel pair wise to the X_{i+i}, Y_{i+1}, Z_{i+1}
- **Position vector** of the origin of the latter in the former.

Let: $\lambda_i = cos\alpha_i, \mu_i = sin\alpha_i$

$$[\mathbf{C}_i]_i \equiv \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\boldsymbol{\Lambda}_i]_{i'} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_i & -\mu_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}$$

THE DH NOTATION



$$[\mathbf{Q}_i]_i = [\mathbf{C}_i]_i[\mathbf{\Lambda}_i]_i = \begin{bmatrix} \cos\theta_i & -\lambda_i \sin\theta_i & \mu_i \sin\theta_i \\ \sin\theta_i & \lambda_i \cos\theta_i & -\mu_i \cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}$$

THE DH NOTATION

One more factoring of matrix Qi, which finds applications in manipulator kinematics, is given below:

$$\mathbf{Q}_i = \mathbf{Z}_i \mathbf{X}_i$$

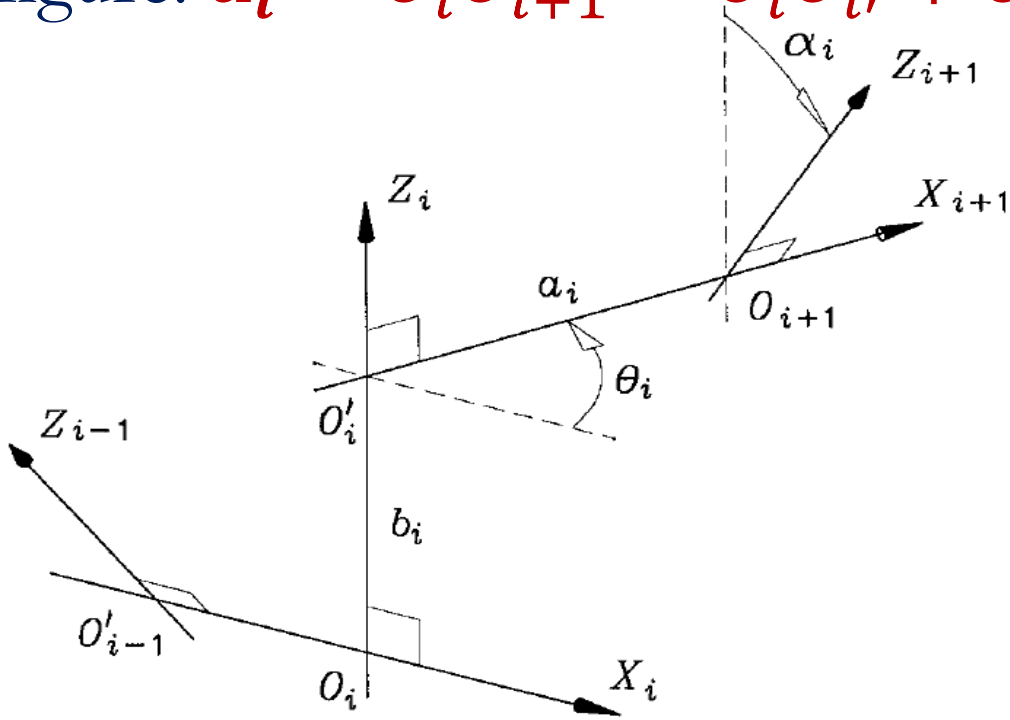
$\mathbf{X}_\mathrm{i}$ and $\mathbf{Z}_\mathrm{i}$ defined as two pure reflections, the former about the Y_iZ_i plane, the latter about the X_iY_i plane

$$[\mathbf{X}_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\lambda_i & \mu_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}; \quad [\mathbf{Z}_i] = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ \sin\theta_i & -\cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{X}_i$ and $\mathbf{Z}_i$ are symmetric and self-inverse

In order to derive an expression for the position vector $\mathbf{a}_i$ connecting the origin O_i of $\mathcal{F}_i$ with $\mathcal{F}_{i+1}$, O_{i+1} reference is made to Figure showing the relative positions of the different origins and axes involved.

From this figure: $\mathbf{a}_i \equiv \overrightarrow{O_i O_{i+1}} = \overrightarrow{O_i O_{i'}} + \overrightarrow{O_{i'} O_{i+1}}$



Where:

$$\left[\overrightarrow{O_i O_{i'}}\right]_i = \begin{bmatrix} 0 \\ 0 \\ b_i \end{bmatrix} \qquad \left[\overrightarrow{O_{i'} O_{i+1}}\right]_{i+1} = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix}$$

The two foregoing vectors should be expressed in the same coordinate frame.

$$\left[\overrightarrow{O_{i'} O_{i+1}}\right]_i = [\mathbf{Q}_i]_i \times \left[\overrightarrow{O_{i'} O_{i+1}}\right]_{i+1} = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ 0 \end{bmatrix}$$

Hence:

$$[\mathbf{a}_i]_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ b_i \end{bmatrix} = \mathbf{Q}_i \mathbf{b}_i \quad ; [\mathbf{b}_i] = \begin{bmatrix} a_i \\ b_i \mu_i \\ b_i \lambda_i \end{bmatrix}$$

THE KINEMATICS OF SIX-REVOLUTE MANIPULATORS

The kinematics of serial manipulators is about studying the:

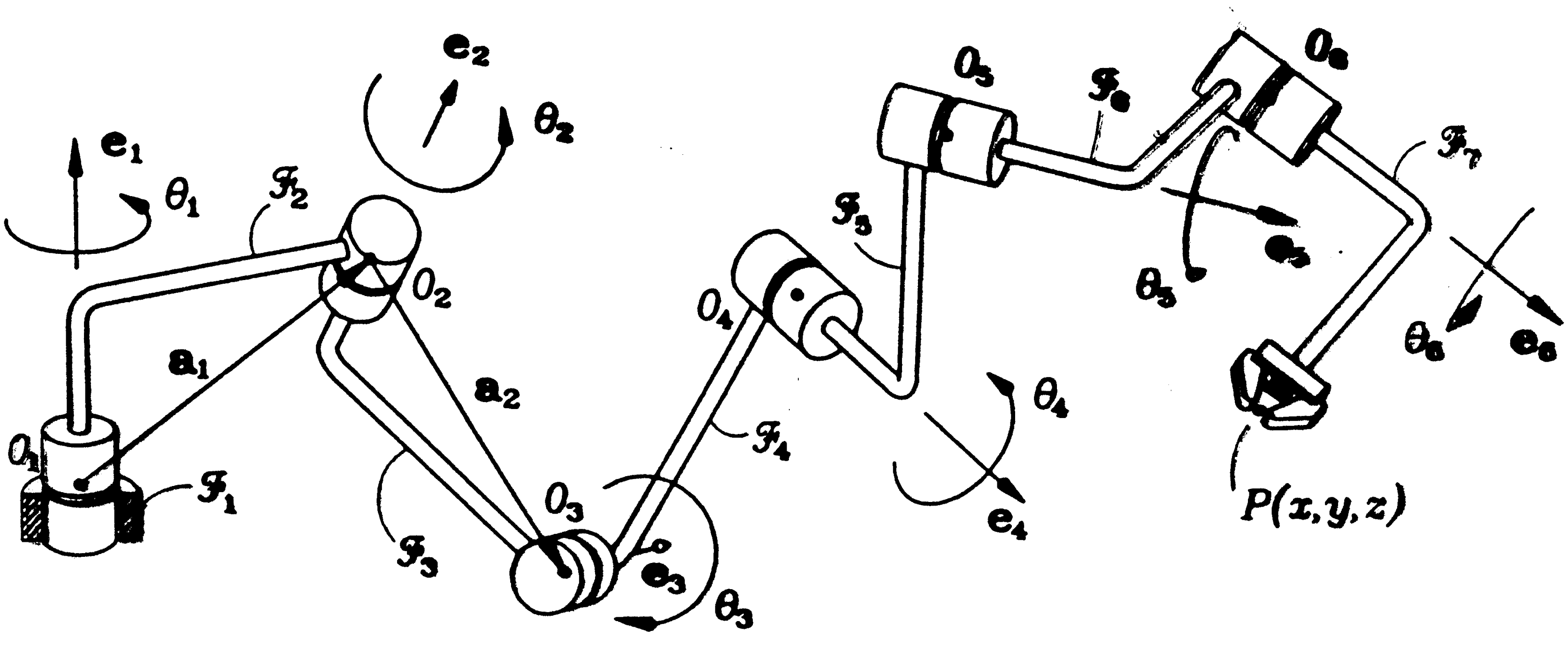
- Geometry: finding relations between *joint variables* and *Cartesian variables*.
- Relation between *joint rates* and the twist of the EE.
- the relations between *joint accelerations* with the time-rate of change of the twist of the EE also pertain to robot kinematics.

THE KINEMATICS OF SIX-REVOLUTE MANIPULATORS

We distinguish two problems:

- **Direct Displacement Problem (DDP)** where the six joint variables of a given six-axis manipulator are assumed to be known and the problem consisting in finding the pose of the EE.
- **Inverse Displacement Problem(IDP)** where, on the contrary, the pose of the EE is given, while the six joint variables that produce this pose are to be found.

SERIAL SIX-AXIS MANIPULATOR



SERIAL SIX-AXIS MANIPULATOR

The relative position and orientation of $\mathcal{F}_{i+1}$ with respect to $\mathcal{F}_i$ is given by matrix **Q***i* and vector $\mathbf{a}_i$, respectively, which are displayed below for quick reference

$$[\mathbf{Q}_i] = \begin{bmatrix} \cos\theta_i & -\lambda_i \sin\theta_i & \mu_i \sin\theta_i \\ \sin\theta_i & \lambda_i \cos\theta_i & -\mu_i \cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} ; \ [\mathbf{a}_i] = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ b_i \end{bmatrix}$$

$\mathbf{Q}_i$ denotes the matrix rotating $\mathcal{F}_i$ into an orientation coincident with that of $\mathcal{F}_{i+1}$.

$\mathbf{a}_i$ denotes the vector joining the origin of F_i with that of F_{i+1}, directed from the former to the latter.

SERIAL SIX-AXIS MANIPULATOR

The orientation **Q** of the EE is obtained as a result of the six individual rotations $\{\mathbf{Q}_i\}_6^1$ about each revolute axis through an angle θ_i, in a *sequential order*, from 1 to 6.

$$[\mathbf{Q}_6]_1[\mathbf{Q}_5]_1[\mathbf{Q}_4]_1[\mathbf{Q}_3]_1[\mathbf{Q}_2]_1[\mathbf{Q}_1]_1 = [\mathbf{Q}]_1$$

$$[\mathbf{a}_1]_1 + [\mathbf{a}_2]_1 + [\mathbf{a}_3]_1 + [\mathbf{a}_4]_1 + [\mathbf{a}_5]_1 + [\mathbf{a}_6]_1 = [\mathbf{p}]_1$$

Both expressed in $\mathcal{F}_1$.

It is hence convenient to represent the foregoing relations in each individual frame, we obtain:

$$\mathrm{Q}_1\mathrm{Q}_2\mathrm{Q}_3\mathrm{Q}_4\mathrm{Q}_5\mathrm{Q}_6 = \mathrm{Q}$$

SERIAL SIX-AXIS MANIPULATOR

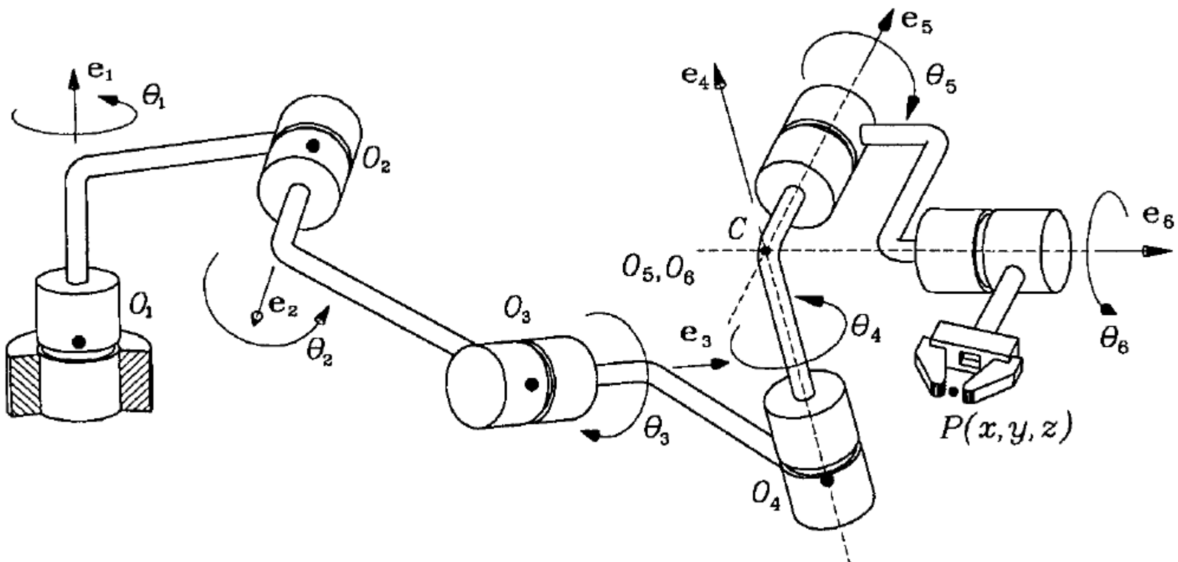
Foregoing equations can be cast in a more compact form if homogeneous transformations are now introduced.

Thus, if we let $\mathbf{T}_i = \{\mathbf{T}_i\}_i$ be the 4 x 4 matrix transforming $\mathcal{F}_{i+1}$coordinates into $\mathcal{F}_i$ -coordinates, the foregoing equations can be written in 4 x 4 matrix form, namely,

$$\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4\mathbf{T}_5\mathbf{T}_6 = \mathbf{T}$$

with **T** denoting the transformation of coordinates from the end-effector frame to the base frame. Thus, **T** contains the pose of the end-effector.

DECOUPLED MANIPOLATORS



Decoupled manipolators are those whose last three joints have intersecting axes.

These joints, then, constitute the **wrist** of the manipulator, which is said to be *spherical*, because when the point of intersection *C* is kept fixed, all the points of the wrist move on spheres centered at *C*.

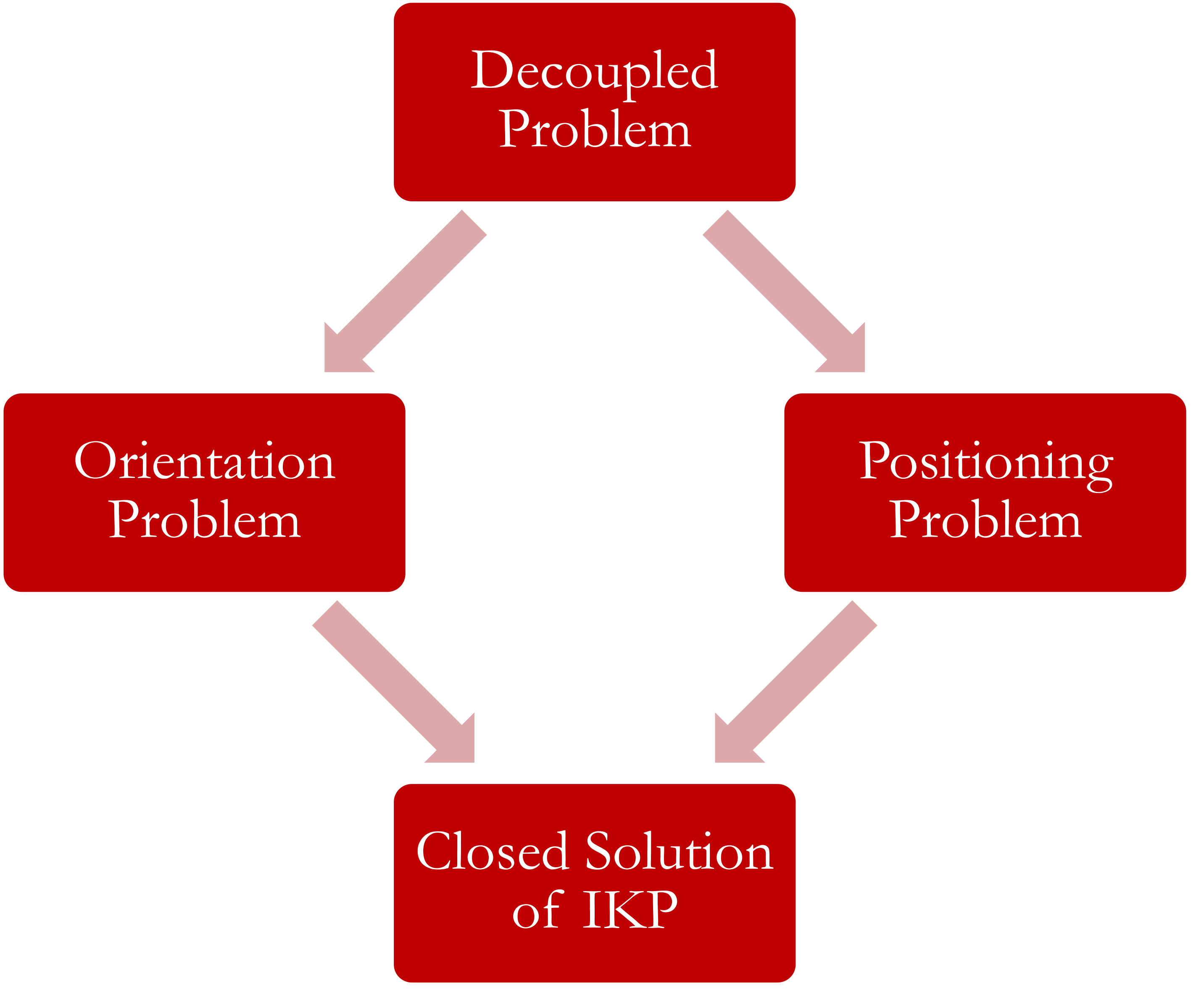
DECOUPLED MANIPOLATORS

In terms of the DH parameters of the manipulator, in a decoupled manipulator:

$$a_4 = a_5 = b_5 = 0$$

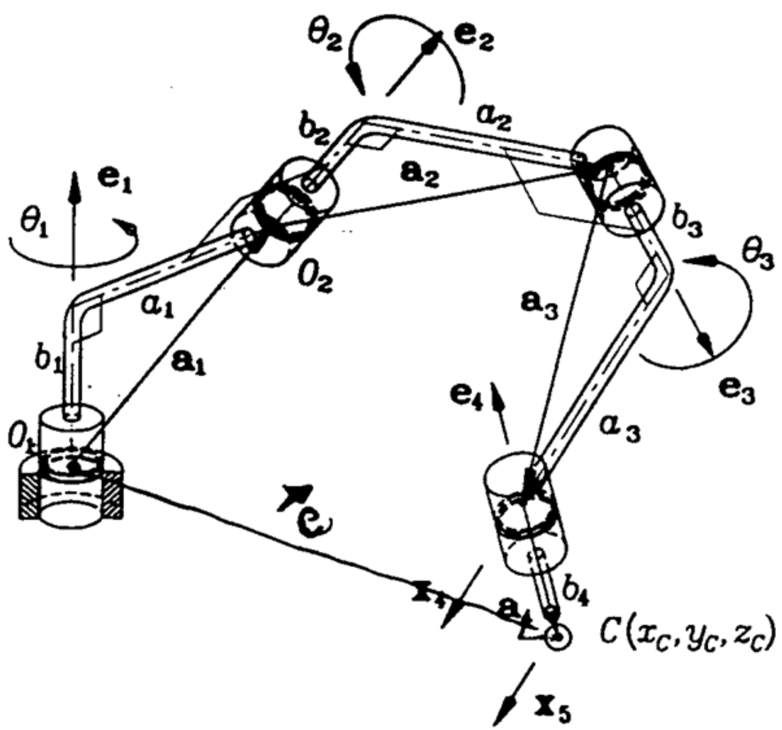
Thus the origins of frames 5 and 6 are coincident.

For decoupled manipulators we conduct the displacement analysis by decoupling the positioning problem from the orientation problem

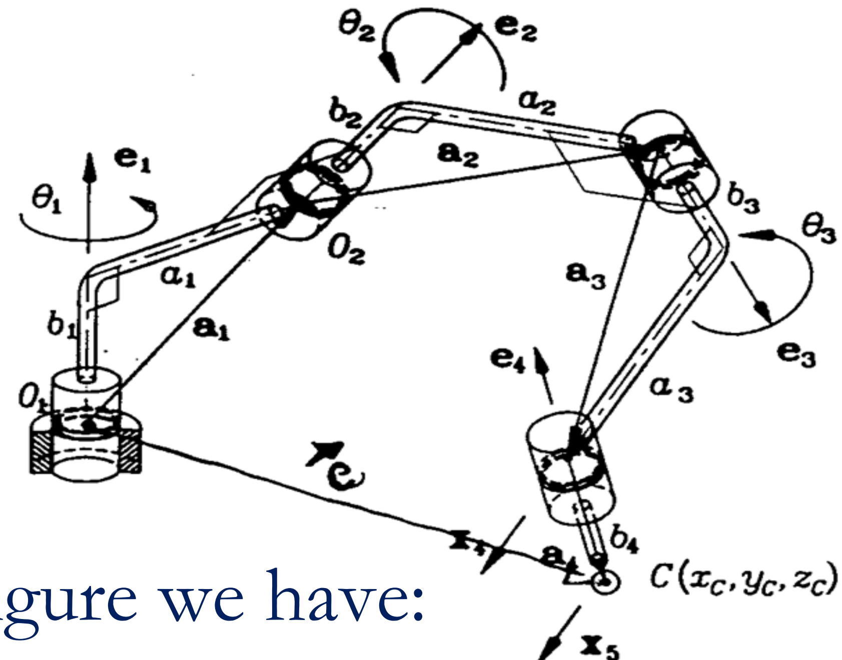

Decoupled Problem
Orientation Problem
Positioning Problem
Closed Solution of IKP

THE POSITIONING PROBLEM

Let C denote the intersection of axes 4, 5, and 6, i.e., the center of the spherical wrist, and let c denote the position vector of this point. The position of C is independent of joint angles θ_4, θ_5, and θ_6; hence, only the first three joints are to be considered for this analysis.



THE POSITIONING PROBLEM



From the figure we have:

$$\mathbf{a}_1 + \mathbf{Q}_1\mathbf{a}_2 + \mathbf{Q}_1\mathbf{Q}_2\mathbf{a}_3 + \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{a}_4 = \mathbf{c}$$

(Expressed in $\mathcal{F}_1$-coordinates)

This equation can be readily rewritten in the form:

$$\mathbf{a}_2 + \mathbf{Q}_2\mathbf{a}_3 + \mathbf{Q}_2\mathbf{Q}_3\mathbf{a}_4 = \mathbf{Q}_1{}^T(\mathbf{c} - \mathbf{a}_1)$$

THE POSITIONING PROBLEM

If we recall that $\mathbf{a}_i = \mathbf{Q}_i \mathbf{b}_i$, the previuos equation becomes:

$$\mathbf{Q}_2(\mathbf{b}_2 + \mathbf{Q}_3\mathbf{b}_3 + \mathbf{Q}_3\mathbf{Q}_4\mathbf{b}_4) = \mathbf{Q}_1{}^T\mathbf{c} - \mathbf{b}_1$$

Since we are dealing with a decoupled manipulator, we have:

$$\mathbf{a}_4 = \mathbf{Q}_4\mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ b_4 \end{bmatrix} = b_4\mathbf{e}$$

Thus the product $Q_3Q_4b_4$ reduces to:

$$\mathbf{Q}_3\mathbf{Q}_4\mathbf{b}_4 = b_4\mathbf{Q}_3\mathbf{e} = b_4\mathbf{u}_3$$

THE POSITIONING PROBLEM

Hence:

$$\mathbf{Q}_2(\mathbf{b}_2+\mathbf{Q}_3\mathbf{b}_3+b_4\mathbf{u}_3)=\mathbf{Q}_1{}^T\mathbf{c}-\mathbf{b}_1 \quad (1)$$

Further, an expression for **c** can be derived in terms of **p**, the position vector of the operation point of the EE, and **Q**, namely,

$$\mathbf{c}=\mathbf{p}-\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{a}_5-\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{a}_6$$

Since:

$a_4=a_5=b_5=0$ and $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6=\mathbf{Q}\mathbf{Q}_6^{\mathbf{T}}$

$$\mathbf{c}=\mathbf{p}-\mathbf{Q}\mathbf{Q}_6{}^T\mathbf{a}_6=\mathbf{p}-\mathbf{Q}\mathbf{b}_6$$

THE POSITIONING PROBLEM

The $\mathcal{F}_1$-components of P and C position vectors are defined as:

$$[\boldsymbol{p}]_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad [\mathbf{c}]_1 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

so $\mathbf{c} = \mathbf{p} - \mathbf{Q}\mathbf{Q}_6{}^T \mathbf{a}_6 = \mathbf{p} - \mathbf{Q}b_6$ can be expanded in the form:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - (q_{11}a_6 + q_{12}b_6\mu_6 + q_{13}b_6\lambda_6) \\ y - (q_{21}a_6 + q_{22}b_6\mu_6 + q_{23}b_6\lambda_6) \\ z - (q_{31}a_6 + q_{32}b_6\mu_6 + q_{33}b_6\lambda_6) \end{bmatrix}$$

THE POSITIONING PROBLEM

In solving the foregoing system of equations, we first note that in this equation

$$\mathbf{Q}_2(\mathbf{b}_2+\mathbf{Q}_3\mathbf{b}_3 + \mathrm{b}_4\mathbf{u}_3) = \mathbf{Q}_1{}^T\mathbf{c} - \mathbf{b}_1 \quad (1)$$

- the left-hand side of this equation appears multiplied by $\mathbf{Q}_2$;
- θ_2 does not appear in in the right-hand side.

This implies that

- if the Euclidean norms of the two sides of that equation are equated, the resulting equation will not contain θ_2
- the third scalar equation of the same equation is independent of θ_2

THE POSITIONING PROBLEM

If we denote by *l* the left-hand of the equation (1) and by *r* its right-hand side, then we have:

$$l^2 \equiv {b_2}^2 + {b_3}^2 + {b_4}^2 + 2\mathbf{b}_2{}^T\mathbf{Q}_3\mathbf{b}_3 + 2b_4\mathbf{b}_2{}^T\mathbf{u}_3 + 2\lambda_3 b_3$$

$$r^2 \equiv \|\mathbf{c}\|^2 + \|\mathbf{b}_1\|^2 - 2\mathbf{b}_1{}^T\mathbf{Q}_1{}^T\mathbf{c}$$

From which it is apparent that l^2 is linear in $\mathbf{x}_3$ and r^2 is linear in $\mathbf{x}_1$.

Upon equating l^2 with r^2, then, an equation linear in $\mathbf{x}_1$ and $\mathbf{x}_3$ is readily derived, namely

$$Ac_1 + Bs_1 + Cc_3 + Ds_3 + E = 0$$

THE POSITIONING PROBLEM

$$Ac_1 + Bs_1 + Cc_3 + Ds_3 + E = 0$$

Whose coefficients do not contain any unknown:

$A = 2a_1 x_c$

$B = 2a_1 y_c$

$C = 2a_2 a_3 - 2b_2 b_4 \mu_2 \mu_3$

$D = 2a_3 b_2 \mu_2 + 2b_2 b_4 \mu_2 \mu_3$

$E = {a_2}^2 + {a_3}^2 + {b_2}^2 + {b_3}^2 + {b_4}^2 - {a_1}^2 - {x_c}^2 - {y_c}^2 - (z_c - b_1)^2 + 2b_2 b_3 \lambda_3 + 2b_2 b_4 \lambda_2 \lambda_3 + 2b_3 b_4 \lambda_3$

THE POSITIONING PROBLEM

Moreover, the third scalar equation of equation (1) takes on the form:

$$Fc_1 + Gs_1 + Hc_3 + Is_3 + J$$

Where:

$$F = y_c\mu_1$$

$$G = -x_c\mu_1$$

$$H = -b_4\mu_2\mu_3$$

$$I = a_3\mu_2$$

$$J = b_2 + b_3\lambda_2 + b_4\lambda_2\lambda_3 - (z_c - b_1)\lambda_1$$

THE POSITIONING PROBLEM

The two above equations, can be solved for c_3 and s_3, namely:

$$c_1 = \frac{-G(Cc_3 + Ds_3 + E) + B(Hc_3 + Is_3 + J)}{\Delta_1} \quad (2)$$

$$s_1 = \frac{F(Cc_3 + Ds_3 + E) - A(Hc_3 + Is_3 + J)}{\Delta_1} \quad (3)$$

Where Δ_1 is defined as:

$$\Delta_1 = AG - FB = -2a_1\mu_1({x_c}^2 + {y_c}^2)$$

THE POSITIONING PROBLEM

Both sides of previous equations are squared, the squares thus obtained are then added, and the sum is equated to 1, which leads to a quadratic equation in $\mathbf{x}_3$, namely:

$$K{c_3}^2 + L{s_3}^2 + Mc_3s_3 + Nc_3 + Ps_3 + Q = 0$$

whose coefficients are given below:

$$K = (4{a_1}^2H^2 + {\mu_1}^2C^2)$$
$$L = (4{a_1}^2I^2 + {\mu_1}^2D^2)$$
$$M = 2(4{a_1}^2HI + {\mu_1}^2CD)$$
$$N = 2(4{a_1}^2HJ + {\mu_1}^2CE)$$
$$P = 2(4{a_1}^2IJ + {\mu_1}^2DE)$$
$$Q = (4{a_1}^2J^2 + {\mu_1}^2E^2 - 4{a_1}^2{\mu_1}^2\rho^2)$$

Where: $\rho^2 = x_c^2 + y_c^2$

THE POSITIONING PROBLEM

Now, two well-known trigonometric identities are introduced, namely,

$$c_3 \equiv \frac{1-\tau_3^2}{1+\tau_3^2} \quad \text{e} \quad s_3 \equiv \frac{2\tau_3}{1+\tau_3^2} \qquad \text{con} \qquad \tau_3 \equiv \tan\left(\frac{\theta_3}{2}\right)$$

Upon substitution of the foregoing identities, a quartic equation in τ_3 is obtained, i.e.

$$R\tau_3^4 + S\tau_3^3 + T\tau_3^2 + U\tau_3 + V = 0$$

THE POSITIONING PROBLEM

After some simplifications, these coefficients take on the forms:

$$R = [4a_1^2(J - H)^2 + \mu_1^2(E - C)^2 - 4\rho^2 a_1^2 \mu_1^2]$$
$$S = 4[4a_1^2 I[(J - H) + \mu_1^2 D(E - C)]$$
$$T = 2[4a_1^2(J^2 - H^2 + 2I^2) + 2\mu_1^2(E^2 - C^2 + 2D^2) - 4\rho^2 a_1^2 \mu_1^2]$$
$$U = 4[4a_1^2 I(H + J) + \mu_1^2 D(C + E)]$$
$$V = [4a_1^2(J + H)^2 + \mu_1^2(E + C)^2 - 4\rho^2 a_1^2 \mu_1^2]$$

Thus, up to four possible values of θ_3 can be obtained, namely,

$$(\theta_3)_i = 2\,arctan[(\tau_3)_i] \qquad i = 1,2,3,4$$

THE POSITIONING PROBLEM

Once the four values of θ_3 are available, each of these is substituted in eq. (2)and (3), which produce four different values of θ_1.

For each value of θ_1 and θ_3, one value of θ_2 can be computed from the first two scalar equations of eq.(1), which are displayed below:

$$A_{11}cos\theta_2 + A_{12}sen\theta_2 = x_c cos\theta_1 + y_c sen\theta_1 - a_1$$
$$A_{12}cos\theta_2 + A_{11}sen\theta_2 = -x_c\lambda_1 sen\theta_1 + y_c\lambda_1 cos\theta_1 + (z_c - b_1)\mu_1$$

Where:

$$A_{11} = a_2 + a_3 cos\theta_3 + b_4\mu_3 sen\theta_3$$
$$A_{12} = -a_3\lambda_2 sen\theta_3 + b_3\mu_2 + b_4\lambda_2\mu_3 cos\theta_3 + b_4\mu_2\lambda_3$$

THE POSITIONING PROBLEM

If A_{11} and A_{12} do not vanish simultaneously, angle θ_2 is readily computed as:

$$\mathrm{Cos}\theta_2 = \frac{1}{\Delta_2}\{A_{11}(x_c \cos\theta_1 + y_c \mathrm{sen}\theta_1 - a_1) - A_{12}[-x_c\lambda_1 \mathrm{sen}\theta_1 + y_c\lambda_1 \cos\theta_1 + (z_c - b_1)\mu_1]\}$$

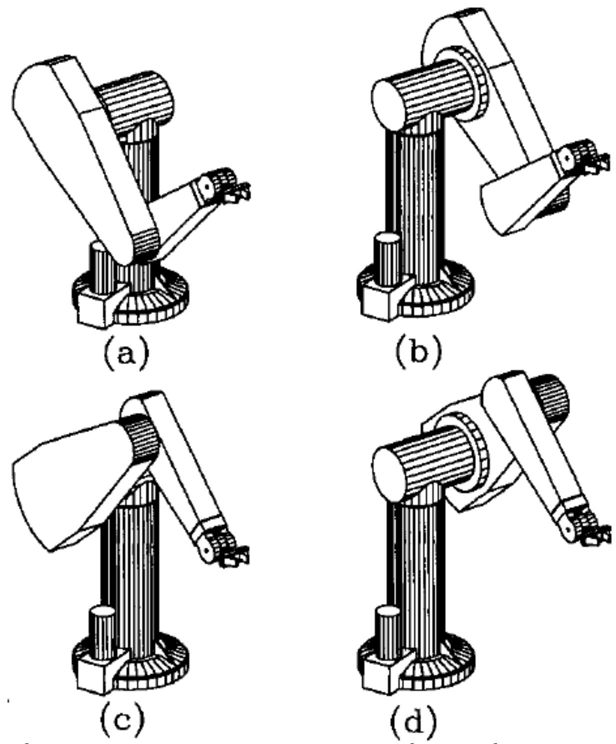
$$\mathrm{Sen}\theta_2 = \frac{1}{\Delta_2}\{A_{12}(x_c \cos\theta_1 + y_c \mathrm{sen}\theta_1 - a_1\ )$$

Where:

$$\Delta_2 = A_{11}^2 + A_{12}^2$$

THE POSITIONING PROBLEM

The four arm configurations for the positioning problem of the **Puma Robot** are:



(a) and (b):elbow down; (c) and (d): elbow up;

(a) and (c):shoulder fore; (b) and (d):shoulder aft.

THE ORIENTATION PROBLEM

This problem consists in determining the wrist angles that will produce a prescribed orientation of the EE. This orientation is given in terms of the rotation matrix Q taking the EE from its home attitude to its current one. Alternatively, the orientation can be given by the natural invariants of the rotation matrix, vector e and angle φ.

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \theta_4 \\ \theta_5 \\ \theta_6 \end{Bmatrix}$$

THE ORIENTATION PROBLEM

Now, since the orientation of the end-effector is given, we know the components of vector $\mathbf{e}_6$ in any coordinate frame. In particular, let

$$[\mathbf{e}_6]_4 = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \text{ and } [\mathbf{e}_5]_4 = \begin{bmatrix} \mu_4 sen\theta_4 \\ -\mu_4 cos\theta_4 \\ \lambda_4 \end{bmatrix}$$

Vectors $\mathbf{e}_5$ and $\mathbf{e}_6$ make an angle α_5, and hence:

$$[\mathbf{e}_6]_4^T[\mathbf{e}_5]_4 = \lambda_5$$

THE ORIENTATION PROBLEM

Upon substitution of $[e_6]_4$ and $[e_5]_4$in the last equation, we obtain:

$$\xi \mu_4 sen\theta_4 - \eta \mu_4 cos\theta_4 + \zeta \lambda_4 = \lambda_5$$

which can be readily transformed, with the aid of the tan-half-angle identities, into a quadratic equation in $\tau_4 = \tan\frac{\theta_4}{2}$

$$(\lambda_5 - \eta\mu_4 - \zeta\lambda_4)\tau_4^2 - 2\xi\mu_4\tau_4 + (\lambda_5 + \eta\mu_4 - \zeta\lambda_4) = 0$$

$$\tau_4 = \xi\mu_4 \pm \frac{\sqrt{(\xi^2 + \eta^2)\mu_4^2 - (\lambda_5 - \zeta\lambda_4)^2}}{\lambda_5 - \zeta\lambda_4 - \eta\mu_4}$$

$\tau 4$ → θ_4

THE ORIENTATION PROBLEM

θ_4 ➡ θ_5

If $\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6 = \mathbf{R}$ with $\mathbf{R}$ defined as: $\mathbf{R} = \mathbf{Q}_3^T\mathbf{Q}_2^T\mathbf{Q}_1^T\boldsymbol{Q}$

Expressions for θ_5 and θ_6 can be readily derived by solving first for $\mathbf{Q}_5$, namely:

$$\mathbf{Q}_5 = \mathbf{Q}_4^T\mathbf{R}\mathbf{Q}_6^T$$

Thus, two equations for θ_5 are obtained by equating the first two components of the third columns of that equation, thereby obtaining:

THE ORIENTATION PROBLEM

Thus, two equations for θ_5 are obtained by equating the first two components of the third columns of that equation, thereby obtaining:

$$\mu_5 s_5 = (\mu_6 r_{12} + \lambda_6 r_{13})c_4 + (\mu_6 r_{22} + \lambda_6 r_{23})s_4$$

$$-\mu_5 c_5 = -\lambda_4(\mu_6 r_{12} + \lambda_6 r_{13})s_4 + \lambda_4(\mu_6 r_{22} + \lambda_6 r_{23})c_4 + \mu_4(\mu_6 r_{32} + \lambda_6 r_{33})$$

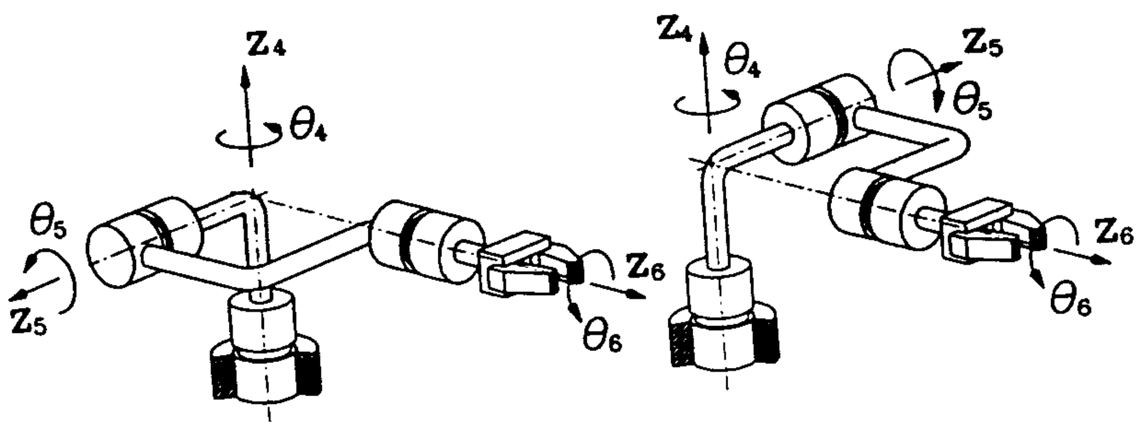
$s_{5,} c_5$ → θ_5

which thus yield a unique value of θ_5 for every value of $\theta_{4,}$

THE ORIENTATION PROBLEM

Finally, with θ_4 and θ_5 known, it is a simple matter to calculate θ_6. This is done upon solving for $\mathbf{Q}_6$ from:

$$\mathbf{Q}_6 = \mathbf{Q}_5^T \mathbf{Q}_4^T \mathbf{R}$$



Hence, c_6 and s_6 are determined as:

$$\begin{cases} c_6 = \omega_1 c_5 + \omega_2 s_5 \\ s_6 = -\omega_1 \lambda_5 s_5 + \omega_2 \lambda_5 c_5 + \omega_3 \mu_5 \end{cases}$$

THE ORIENTATION PROBLEM

When combined with the four postures of a decoupled manipulator leading to one and the same location of its wrist center— positioning problem — a maximum of eight possible combinations of joint angles for a single pose of the end-effector of a decoupled manipulator are found