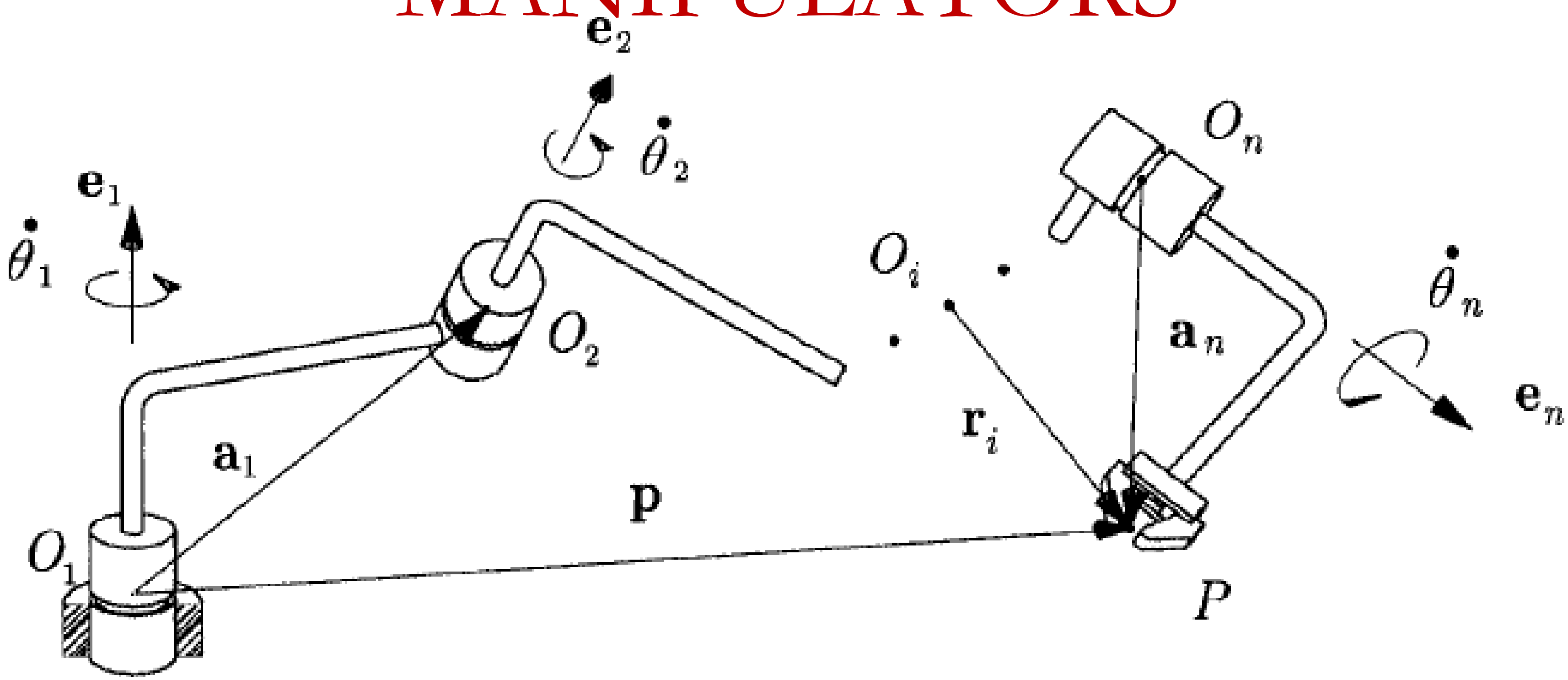
KINETOSTATICS OF SERIAL ROBOTS

VELOCITY ANALYSIS OF SERIAL MANIPULATORS



We consider the manipulator of Figure in which a joint coordinate θ_i, a joint rate $\dot{\theta}_i$, and a unit vector $\mathbf{e}_i$ are associated with each revolute axis.

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

The X_i, Y_i, Z_i coordinate frame, attached to the $(i-1)$st link, is not shown, but its origin O_i is indicated.

If the angular-velocity vector of the ith link is denoted by ω_i then we have

$$\boldsymbol{\omega}_0 = \mathbf{0}$$

$$\boldsymbol{\omega}_1 = \dot{\theta}_1 \mathbf{e}_1$$

$$\boldsymbol{\omega}_2 = \dot{\theta}_1 \mathbf{e}_1 + \dot{\theta}_2 \mathbf{e}_2$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$\boldsymbol{\omega}_n = \dot{\theta}_1 \mathbf{e}_1 + \dot{\theta}_2 \mathbf{e}_2 + \cdots + \dot{\theta}_n \mathbf{e}_n$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

If the angular velocity of the EE is denoted by $\boldsymbol{\omega}$ then:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n = \dot{\theta}_1\mathbf{e}_1 + \dot{\theta}_2\mathbf{e}_2 + \cdots + \dot{\theta}_n\mathbf{e}_n = \sum_1^n \dot{\theta}_i\mathbf{e}_i$$

Likewise one readily derives

$$\mathbf{p} = \mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n$$

where $\mathbf{p}$ denotes the position vector of point P of the EE. Upon differentiating both sides of equation, we have:

$$\dot{\mathbf{p}} = \dot{\mathbf{a}}_1 + \dot{\mathbf{a}}_2 + \cdots + \dot{\mathbf{a}}_n$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

Considering that $\dot{\mathbf{a}}_i = \boldsymbol{\omega}_i \times \mathbf{a}_i$

We can rearrange the foregoing equation as:

$$\dot{\mathbf{p}} = \sum_{i}^{n} \dot{\theta}_i \mathbf{e}_i \times \mathbf{r}_i$$

Where vector $\mathbf{r}_i$ is defined as that joining O_i with P, directed from the former to the latter, as:

$$\mathbf{r}_i \equiv \mathbf{a}_i + \mathbf{a}_{i+1} + \cdots + \mathbf{a}_n$$

Let **A** and **B** denote the 3xn matrices defined as:

$$\boldsymbol{A} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]$$

$$\mathbf{B} = [\mathbf{e}_1 \times \mathbf{r}_1 \quad \mathbf{e}_2 \times \mathbf{r}_2 \quad \cdots \quad \mathbf{e}_n \times \mathbf{r}_n]$$

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Furthermore, the *n*-dimensional joint-rate vector $\dot{\boldsymbol{\theta}}$ is defined as:

$$\dot{\boldsymbol{\theta}} \equiv \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \cdots \\ \dot{\theta}_n \end{bmatrix}$$

Thus, $\boldsymbol{\omega}$ and $\dot{\mathrm{p}}$ can be expressed in a more compact form as:

$$\boldsymbol{\omega} = \boldsymbol{A}\dot{\boldsymbol{\theta}} \qquad \dot{\mathbf{p}} = \boldsymbol{B}\dot{\boldsymbol{\theta}}$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

The twist of the EE being defined as:

$$\mathbf{t} = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}$$

The EE twist is thus nearly related to the joint-rate vector $\dot{\boldsymbol{\theta}}$ as: $$J\dot{\boldsymbol{\theta}} = \mathbf{t}$$

where $\mathbf{J}$ is the Jacobian matrix defined as the 6xn matrix shown below

$$\mathbf{J} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$$

Moreover, if $\mathbf{j}_i$ denotes the ith column of $\mathbf{J}$, one has

$$\mathbf{j}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_i \times \mathbf{r}_i \end{bmatrix}$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

Vector $\mathbf{a}_i$ joining the origins of the ith and $(i+1)$st frames is no longer of constant magnitude but undergoes a change of magnitude along the axis of the prismatic pair.

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1}$$
$$\dot{\mathbf{a}}_i = \boldsymbol{\omega}_{i-1} \times \mathbf{a}_i + \dot{b}_i \mathbf{e}_i$$

One can readily prove, in this case, that

$$\boldsymbol{\omega} = \dot{\theta}_1 \mathbf{e}_1 + \dot{\theta}_2 \mathbf{e}_2 + \cdots + \dot{\theta}_{i-1} \mathbf{e}_{i-1} + \dot{\theta}_{i+1} \mathbf{e}_{i+1} + \cdots + \dot{\theta}_n \mathbf{e}_n$$
$$\dot{\mathbf{p}} = \dot{\theta}_1 \mathbf{e}_1 \times \mathbf{r}_1 + \cdots + \dot{\theta}_{i-1} \mathbf{e}_{i-1} \times \mathbf{r}_{i-1} + \dot{b}_i \mathbf{e}_i + \dot{\theta}_{i+1} \mathbf{e}_{i+1} \times \mathbf{r}_{i+1} + \cdots + \dot{\theta}_n \mathbf{e}_n \times \mathbf{r}_n$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

From which it is apparent that the relation between the twist of the EE and the joint-rate vector is formally identical to that appearing in $\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{t}$ if vector $\dot{\boldsymbol{\theta}}$ is defined as:

$$\dot{\boldsymbol{\theta}} = [\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_{i-1}, \dot{b}_i, \dot{\theta}_{i+1}, \ldots, \dot{\theta}_n]^T$$

and the *i*th column of $\mathbf{J}$ changes to:

$$\mathbf{j}_i = \begin{bmatrix} 0 \\ \mathbf{e}_i \end{bmatrix}$$

In particular, for six-axis manipulators, $\mathbf{J}$ is a 6×6 matrix.

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

Whenever this matrix is nonsingular, can be solved for $\dot{\boldsymbol{\theta}}$, namely,

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}^{-1}\mathbf{t}$$

$\dot{\boldsymbol{\theta}}$ is computed using a numerical procedure. The most suitable is the Gauss-elimination algorithm known as LU decomposition.

Gaussian elimination produces the solution by recognizing that system of linear equations in most easily solved when it is in either upper or lower triangular form.

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

Matrix **J** is factored into the unique **L** and **U** factors in the form:

$$\mathbf{J} = \mathbf{LU}$$

Where **L** is the lower and **U** is the upper triangular. Moreover, they have the forms:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ e_{21} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ e_{n1} & e_{n2} & \dots & 1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 1 & e_{12} & \dots & e_n \\ 0 & e_{22} & \dots & e_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Thus, the unknown vector of joint rates can now be computed from two triangular systems, namely,

$$\mathbf{Ly} = \mathbf{t} \qquad \mathbf{U}\dot{\boldsymbol{\theta}} = \boldsymbol{y}$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

The latter equations are then solved, first for $\mathbf{y}$ and then for $\dot{\boldsymbol{\theta}}$, by application of only forward and backward substitutions, respectively

$$\mathbf{y} = \mathbf{L}^{-1}\mathbf{t} \qquad \dot{\boldsymbol{\theta}} = \mathbf{U}^{-1}\mathbf{y}$$

Thus, the solution of a system of n linear equations in n unknowns, using the LU-decomposition method, can be accomplished with M_n multiplications and A_n additions, as given below

$$M_n = \frac{n}{6}(2n^2 + 9n + 1) \qquad A_n = \frac{n}{3}(n^2 + 3n - 4)$$

VELOCITY ANALYSIS OF SERIAL MANIPULATORS

Hence, the velocity resolution of a six-axis manipulator of arbitrary architecture requires M_6 multiplications and A_6 additions, as given below:

$$M_6 = 127 \qquad A_6 = 100$$

DECOUPLED MANIPULATORS

For manipulators with this type of architecture, it is more convenient to deal with the velocity of the center *C* of the wrist than with that of the operation point *P*.

$$\mathbf{t}_C = \mathbf{J}\,\dot{\boldsymbol{\theta}}$$

where $\mathbf{t}_C$ is defined as: $\mathbf{t}_C = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{c}} \end{bmatrix}$

and can be obtained from $\mathbf{t}_P = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}$ using the twist-transfer formula:

$$\mathbf{t}_C = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{P} - \mathbf{C} & \mathbf{1} \end{bmatrix} \mathbf{t}_P$$

DECOUPLED MANIPULATORS

With **C** and **P** defined as the cross-product matrices of the position vectors **c** and **p**, respectively.

If in general, $\mathbf{J}_A$ denotes the Jacobian defined for a point A of the EE and $\mathbf{J}_B$ that defined for another point B, then the relation between $\mathbf{J}_A$ and $\mathbf{J}_B$ is:

$$\mathbf{J}_B = \mathbf{U}\mathbf{J}_A$$

where the 6×6 matrix **U** is defined as:

$$\mathbf{U} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{A} - \mathbf{B} & \mathbf{1} \end{bmatrix}$$

Then: $$\det(\mathbf{J}_B) = \det(\mathbf{J}_A)$$

So we have proven that the determinat of the Jacobian matrix of a six-axis manipulator is not affected under a change of operation point of the EE.

DECOUPLED MANIPULATORS

Since C is on the last three joint axes, its velocity is not affected by the motion of the last three joints, and we can write:

$$\dot{\mathbf{c}} = \dot{\theta}_1 \mathbf{e}_1 \times \mathbf{r}_1 + \dot{\theta}_2 \mathbf{e}_2 \times \mathbf{r}_2 + \dot{\theta}_3 \mathbf{e}_3 \times \mathbf{r}_3$$

$$\boldsymbol{\omega} = \dot{\theta}_1 \mathbf{e}_1 + \dot{\theta}_2 \mathbf{e}_2 + \dot{\theta}_3 \mathbf{e}_3 + \dot{\theta}_4 \mathbf{e}_4 + \dot{\theta}_5 \mathbf{e}_5 + \dot{\theta}_6 \mathbf{e}_6$$

the Jacobian takes on the following simple form:

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{O} \end{bmatrix}$$

Where:

$$\mathbf{J}_{11} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

$$\mathbf{J}_{12} = [\mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6]$$

$$\mathbf{J}_{21} = [\mathbf{e}_1 \times \mathbf{r}_1 \quad \mathbf{e}_2 \times \mathbf{r}_2 \quad \mathbf{e}_3 \times \mathbf{r}_3]$$

DECOUPLED MANIPULATORS

Further, vector $\dot{\boldsymbol{\theta}}$ is partitioned accordingly:

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_a \\ \dot{\boldsymbol{\theta}}_w \end{bmatrix}$$

Where:

$$\dot{\boldsymbol{\theta}}_a = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \qquad \dot{\boldsymbol{\theta}}_w = \begin{bmatrix} \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}$$

Henceforth, the three components of $\dot{\boldsymbol{\theta}}_a$ will be referred to as the arm rates, whereas those of $\dot{\boldsymbol{\theta}}_w$ will be called the wrist rates.

DECOUPLED MANIPULATORS

$$\begin{cases} \mathbf{J}_{11}\dot{\boldsymbol{\theta}}_a + \mathbf{J}_{12}\dot{\boldsymbol{\theta}}_w = \boldsymbol{\omega} \\ \mathbf{J}_{21}\dot{\boldsymbol{\theta}}_a = \dot{\mathbf{c}} \end{cases}$$

$$\begin{cases} \mathbf{J}_{21}\dot{\theta}_a = \dot{\mathbf{c}} \\ \mathbf{J}_{12}\dot{\boldsymbol{\theta}}_w = \boldsymbol{\omega} - \mathbf{J}_{11}\dot{\boldsymbol{\theta}}_a \end{cases} \Longrightarrow \begin{cases} \dot{\boldsymbol{\theta}}_a = \mathbf{J}_{21}^{-1}\dot{\mathbf{c}} \\ \dot{\boldsymbol{\theta}}_w = \mathbf{J}_{12}^{-1}(\boldsymbol{\omega} - \boldsymbol{J}_{11}\dot{\boldsymbol{\theta}}_a) \end{cases}$$

Now, if we recall the concept of reciprocal bases, the above inverses can be represented explicitly:

$$\Delta_{21} = \det(\mathbf{J}_{21}) = (\mathbf{e}_1 \times \mathbf{r}_1) \times (\mathbf{e}_2 \times \mathbf{r}_2) \times (\mathbf{e}_3 \times \mathbf{r}_3)$$

$$\Delta_{12} = \mathbf{e}_4 \times \mathbf{e}_5 \times \mathbf{e}_6$$

DECOUPLED MANIPULATORS

Then:

$$J_{21}^{-1} = \frac{1}{\Delta_{21}} \begin{bmatrix} [(\mathbf{e}_2 \times \mathbf{r}_2) \times (\mathbf{e}_3 \times \mathbf{r}_3)]^T \\ [(\mathbf{e}_3 \times \mathbf{r}_3) \times (\mathbf{e}_1 \times \mathbf{r}_1)]^T \\ [(\mathbf{e}_1 \times \mathbf{r}_1) \times (\mathbf{e}_2 \times \mathbf{r}_2)]^T \end{bmatrix} \qquad J_{12}^{-1} = \frac{1}{\Delta_{12}} \begin{bmatrix} (\mathbf{e}_5 \times \mathbf{e}_6)^T \\ (\mathbf{e}_6 \times \mathbf{e}_4)^T \\ (\mathbf{e}_4 \times \mathbf{e}_5)^T \end{bmatrix}$$

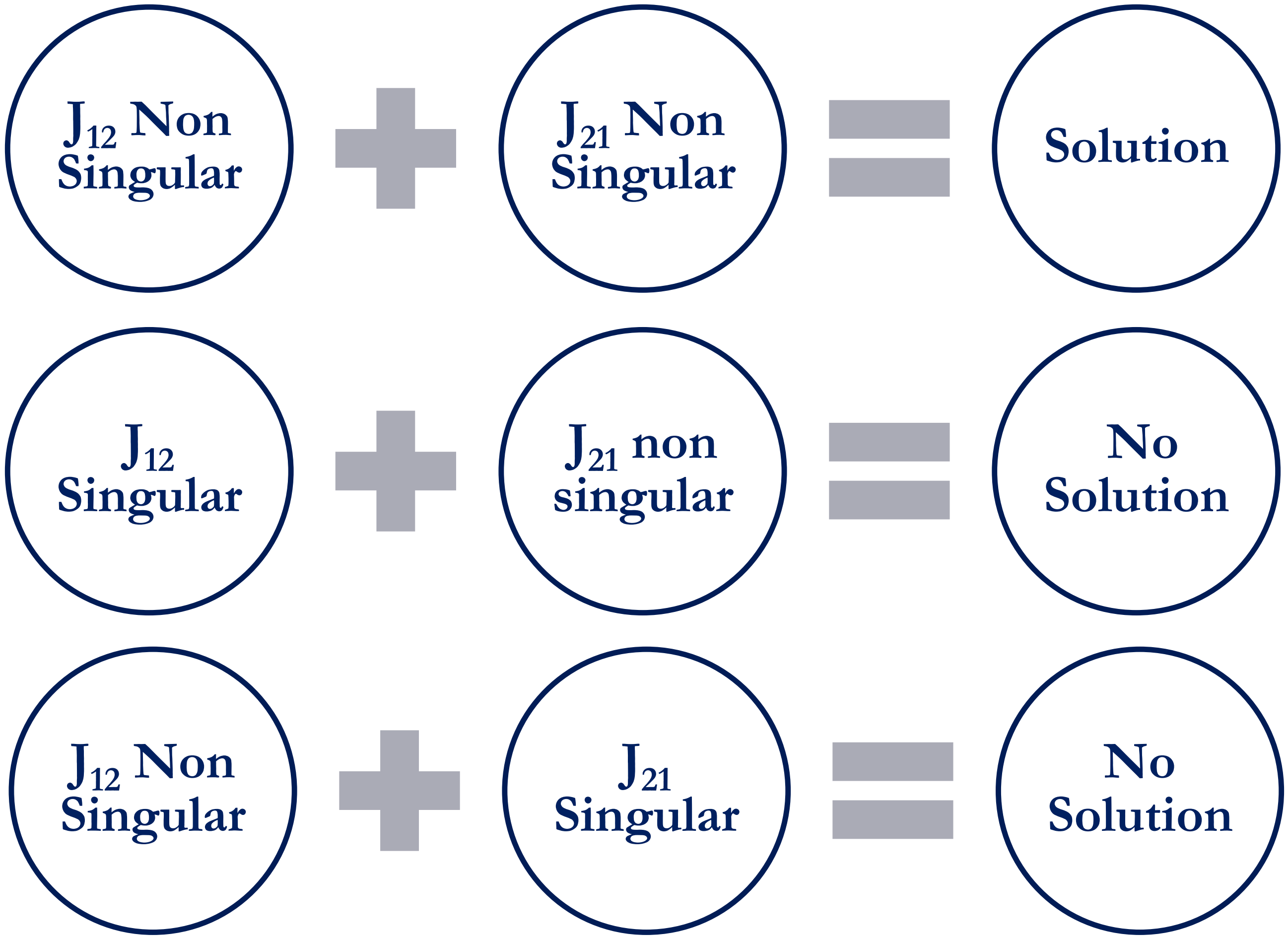
Therefore:

$$\dot{\boldsymbol{\theta}}_a = \frac{1}{\Delta_{21}} \begin{bmatrix} (\mathbf{e}_2 \times \mathbf{r}_2) \times (\mathbf{e}_3 \times \mathbf{r}_3) \cdot \dot{\mathbf{c}} \\ (\mathbf{e}_3 \times \mathbf{r}_3) \times (\mathbf{e}_1 \times \mathbf{r}_1) \cdot \dot{\mathbf{c}} \\ (\mathbf{e}_1 \times \mathbf{r}_1) \times (\mathbf{e}_2 \times \mathbf{r}_2) \cdot \dot{\mathbf{c}} \end{bmatrix}$$

and if we let: $\bar{\boldsymbol{\omega}} = \boldsymbol{\omega} - \mathbf{J}_{11}\dot{\theta}_a$

$$\dot{\boldsymbol{\theta}}_w = \frac{1}{\Delta_{21}} \begin{bmatrix} \mathbf{e}_5 \times \mathbf{e}_6 \cdot \bar{\boldsymbol{\omega}} \\ \mathbf{e}_6 \times \mathbf{e}_4 \cdot \bar{\boldsymbol{\omega}} \\ \mathbf{e}_4 \times \mathbf{e}_5 \cdot \bar{\boldsymbol{\omega}} \end{bmatrix}$$

SINGULARITY ANALYSIS OF DECOUPLED MANIPULATORS



ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

Deriving $\mathbf{J}\dot{\boldsymbol{\theta}} = \boldsymbol{t}$, we have:

$$\mathbf{J}\,\ddot{\boldsymbol{\theta}} = \dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\boldsymbol{\theta}}$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{J}^{-1}(\dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\boldsymbol{\theta}})$$

It is apparent that the joint-acceleration vector is computed in exactly the same way as the joint-rate vector. In fact, the LU decomposition of **J** is the same in this case and hence, need not be recomputed. All that is needed is the solution of a lower- and an upper-triangular system, namely,

$$\mathbf{Lz} = \dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\theta} \qquad\qquad \mathbf{U}\ddot{\boldsymbol{\theta}} = \boldsymbol{z}$$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

The two foregoing systems are solved first for $\mathbf{z}$ and then for $\ddot{\boldsymbol{\theta}}$ by forward and backward substitution, respectively.

Thus, the total numbers of multiplications M_t and additions A_t that the forward and backward solutions of the aforementioned systems require are:

$$M_t = n^2 \qquad\qquad A_t = n(n-1)$$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

$$\mathbf{J}\,\ddot{\boldsymbol{\theta}} = \dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\boldsymbol{\theta}}$$

The right-hand side comprises two terms, the first being the specified time-rate of change of the twist of the EE, or twist-rate, for brevity, which is readily available. The second term is not available and must be computed. This term involves the product of the time-derivative of **J** times the previously computed joint-rate vector. Hence, in order to evaluate the right-hand side of that equation, all that is further required is $\dot{\boldsymbol{J}}$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

$$\dot{\mathbf{j}} = \begin{bmatrix} \dot{\mathbf{A}} \\ \dot{\mathbf{B}} \end{bmatrix}$$

Where:

$$\dot{\mathbf{A}} = [\dot{\mathbf{e}}_1 \quad \dot{\mathbf{e}}_2 \quad \cdots \quad \dot{\mathbf{e}}_n]$$
$$\dot{\mathbf{B}} = [\dot{\mathbf{u}}_1 \quad \dot{\mathbf{u}}_2 \quad \cdots \quad \dot{\mathbf{u}}_n]$$

and $\mathbf{u}_i$ denotes $\mathbf{e}_i \times \mathbf{r}_i$, for $i = 1, 2, \ldots, n$. Moreover:

$$\dot{\mathbf{e}}_1 = \boldsymbol{\omega}_0 \times \mathbf{e}_1 = \mathbf{0}$$
$$\dot{\mathbf{e}}_i = \boldsymbol{\omega}_{i-1} \times \mathbf{e}_i = \boldsymbol{\omega}_i \times \mathbf{e}_i$$
$$\dot{\mathbf{u}}_i = \dot{\mathbf{e}}_i \times \mathbf{r}_i + \mathbf{e}_i \times \dot{\mathbf{r}}_i$$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

Differentiating we obtain:

$$\dot{\mathbf{r}}_i = \dot{\mathbf{a}}_i + \dot{\mathbf{a}}_{i+1} + \cdots + \dot{\mathbf{a}}_n$$

Recalling the above equation reduces to

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega}_i \times \mathbf{a}_i + \boldsymbol{\omega}_{i+1} \times \mathbf{a}_{i+1} + \cdots + \boldsymbol{\omega}_n \times \mathbf{a}_n$$

Substitution of equations leads to:

$$\dot{\mathbf{A}} = [\,\mathbf{0} \quad \boldsymbol{\omega}_1 \times \mathbf{e}_2 \quad \cdots \quad \boldsymbol{\omega}_{n-1} \times \mathbf{e}_n]$$

$$\dot{\mathbf{B}} = [\mathbf{e}_1 \times \dot{\mathbf{r}}_1 \quad \boldsymbol{\omega}_{12} \times \mathbf{r}_2 + \boldsymbol{e}_2 \times \dot{\mathbf{r}}_2 \quad \cdots \quad \boldsymbol{\omega}_{n-1,n} \times \mathbf{r}_n + \mathbf{e}_n \times \dot{\mathbf{r}}_n]$$

With $\dot{\mathbf{r}}_k$ and $\boldsymbol{\omega}_{k,k+1}$ defined as:

$$\dot{\mathbf{r}}_k = \sum_{k}^{n} \boldsymbol{\omega}_i \times \mathbf{a}_i \qquad \boldsymbol{\omega}_{k,k+1} = \boldsymbol{\omega}_k \times \mathbf{e}_{k+1}$$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

The foregoing expressions are invariant and hence, valid in any coordinate frame. All columns of both $\dot{\mathbf{A}}$ and $\dot{\mathbf{B}}$ will have to be represented in the samecoordinate frame. Hence, coordinate transformations will have to be introduced in the foregoing matrix columns in order to have all of these represented in the same coordinate frame,

$$\dot{\mathbf{j}}\dot{\boldsymbol{\theta}} = \dot{\theta}_1 \begin{bmatrix} \mathbf{0} \\ \dot{\mathbf{u}}_1 \end{bmatrix} + \dot{\theta}_2 \begin{bmatrix} \dot{\mathbf{e}}_2 \\ \dot{\mathbf{u}}_2 \end{bmatrix} + \cdots + \dot{\theta}_n \begin{bmatrix} \dot{\mathbf{e}}_n \\ \dot{\mathbf{u}}_n \end{bmatrix}$$

ACCELERATION ANALYSIS OF SERIAL MANIPULATORS

Thus, the total numbers of multiplications and additions required to compute $\dot{\mathbf{J}}\dot{\boldsymbol{\theta}}$ in frame $\mathcal{F}_1$, denoted by M_J and A_J, respectively, are as shown below

$$M_J = 47n - 37 \qquad A_J = 31n - 28$$

The total numbers of multiplications and additions needed to compute the aforementioned right-hand side, denoted by M_r and A_r are:

$$M_r = 47n - 37 \qquad A_r = 31n - 22$$

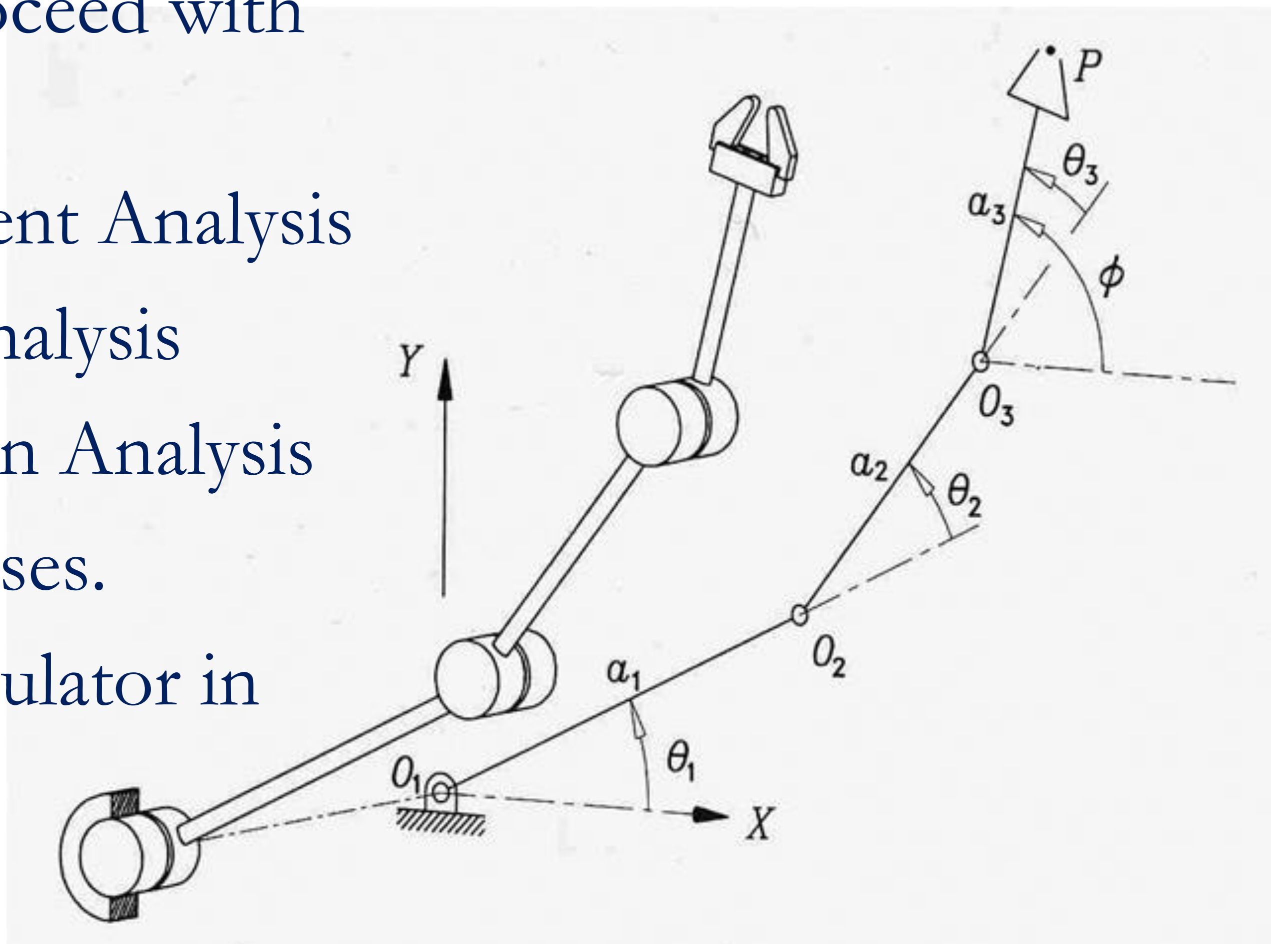
The numbers of multiplications and additions required for an acceleration resolution of a 6-R manipulator of arbitrary architecture are: $M_a = 281 \qquad A_a = 194$

PLANAR MANIPULATORS

Below we proceed with the

- Displacement Analysis
- Velocity Analysis
- Acceleration Analysis
- Static analyses.

Of the manipulator in the figure



PLANAR MANIPULATORS

Let E be defined as an orthogonal matrix rotating 2-D vectors through an angle of 90° counterclockwise. Hence:

$$\mathbf{E} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We thus have

- $\mathbf{E}^{-1} = \mathbf{E}^{T} = -\mathbf{E}$
- $\mathbf{E}^{2} = -\mathbf{1}$

where **1** is the 2x2 identity matrix.

DISPLACEMENT ANALYSIS

The inverse kinematics of the manipulator at hand now consists of determining the values of angles θ_i, for $i = 1, 2, 3$, that will place the end-effector so that its operation point P will be positioned at the prescribed Cartesian coordinates x, y and be oriented at a given angle ϕ with the X axis.

We now have, from the geometry of Figure:

$$a_1 c_1 + a_2 c_{12} = x_c$$

$$a_1 s_1 + a_2 s_{12} = y_c$$

where x_c and y_c denote the Cartesian coordinates of point O_3,

$c_1 = \cos\theta_1$ $\quad$ $c_{12} = \cos(\theta_1 + \theta_2)$ and

$s_1 = sen\,(\theta_1)$ $\quad$ $s_{12} = sen(\theta_1 + \theta_2)$

DISPLACEMENT ANALYSIS

Indeed, from the two foregoing equations we can eliminate both c_{12} and s_{12} by solving for the second terms of the left-hand sides of those equations, namely

$$a_2 c_{12} = x_c - a_1 c_1$$
$$a_2 s_{12} = y_c - a_1 s_1$$

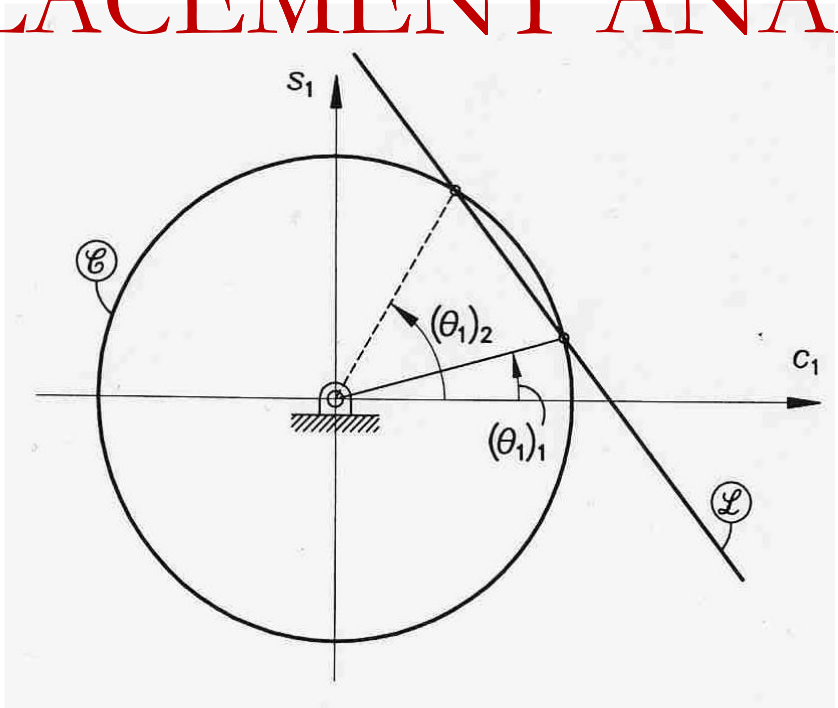
We obtain, after simplification, a linear equation in c_1 and s_1 that represents a line $\mathcal{L}$ in the c_1-s_1 plane:

$$\mathcal{L}: -a_1^2 + a_2^2 + 2a_1 x_c c_1 + 2a_1 y_c s_1 - (x_c^2 + y_c^2) = 0$$

Clearly, the two foregoing variables are constrained by a quadratic equation defining a circle C in the same plane:

$$c_1^2 + s_1^2 = 1$$

DISPLACEMENT ANALYSIS



The real roots of interest are then obtained as the intersections of $\mathcal{L}$ and C. Thus, the problem can admit:

- Two real and distinct roots, if the line and the circle intersect;
- One repeated root if the line is tangent to the circle;
- No real root if the line does not intersect the circle

DISPLACEMENT ANALYSIS

With c_1 and s_1 known, angle θ_1 is fully determined. Note that the two real intersections of $\mathcal{L}$ with C provide each one value of θ_1, as depicted in Figure.

θ_2 is derived from θ_1 as follows:

$$(\theta_2)_i = \tan^{-1}\frac{y_c - a_1(s_1)_i}{x_c - a_1(c_1)_i} - (\theta_1)_i$$

Once θ_1 and θ_2 are available, θ_3 is readily derived from the geometry:

$$\theta_3 = \phi - (\theta_1 + \theta_2)$$

Hence, each pair of $(\theta_1\ \theta_2)$ values yields one single value for θ_3. Since we have two such pairs, the problem admits two real solutions

VELOCITY ANALYSIS

The velocity relation adapted to planar manipulators are:

$$\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{t}$$

Where:

$$J \equiv \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_2 & \mathbf{e}_3 \times \mathbf{r}_3 \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} \equiv \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}, \quad \mathbf{t} \equiv \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}$$

And where and $\{\mathbf{r}_i\}$ are defined as the vectors directed from O_i to P:

$$\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} \qquad i = 1,2,3$$

VELOCITY ANALYSIS

We assume here that the manipulator moves in the *X*-*Y* plane, and hence:

$$\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And with **t** reducing to :

$$\mathbf{t} = \begin{bmatrix} 0 & 0 & \dot{\phi} & \dot{x}_p & \dot{y}_p & 0 \end{bmatrix}^T$$

in which $\dot{x}_p$ and $\dot{y}_p$ denote the components of the velocity of *P*.

VELOCITY ANALYSIS

Thus $\mathbf{e}_\mathrm{i} \times \mathbf{r}_i = \begin{bmatrix} -y_i \\ x_i \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}\,\mathbf{s}_i \\ 0 \end{bmatrix}$

The equation $\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{t}$ reduces to :

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 \\ \mathbf{E}\mathbf{s}_1 & \mathbf{E}\mathbf{s}_2 & \mathbf{E}\mathbf{s}_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \dot{\phi} \\ \dot{\mathbf{p}} \\ 0 \end{bmatrix}$$

Where: $\dot{\mathrm{p}} = [\dot{x} \quad \dot{y}]^T$

Multiplying the first row for $\mathbf{E}\mathbf{s}_1$ and subtracting the latter to the second:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathbf{E}(\mathbf{s}_2 - \mathbf{s}_1) & \mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\mathbf{p}} - \dot{\phi}\mathbf{E}\mathbf{s}_1 \end{bmatrix}$$

VELOCITY ANALYSIS

This system can be reduced to only two equations

$$[\mathbf{E}(\mathbf{s}_2 - \mathbf{s}_1) \quad \mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)] \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \dot{\mathbf{p}} - \dot{\phi}\mathbf{E}\mathbf{s}_1$$

From these equations we can get the two unknowns:

$$\dot{\theta}_2 = \frac{(\mathbf{s}_3 - \mathbf{s}_1)^T\left(\dot{\mathbf{p}} - \dot{\phi}\mathbf{E}\mathbf{s}_1\right)}{(\mathbf{s}_2 - \mathbf{s}_1)^T\mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)}$$

$$\dot{\theta}_3 = \frac{(\mathbf{s}_2 - \mathbf{s}_1)^T\left(\dot{\mathbf{p}} - \dot{\phi}\mathbf{E}\mathbf{s}_1\right)}{(\mathbf{s}_2 - \mathbf{s}_1)^T\mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)}$$

$$\dot{\theta}_1 = \dot{\phi} - (\dot{\theta}_1 + \dot{\theta}_2)$$

ACCELERATION ANALYSIS

Differentiating the equation $\mathbf{J}\dot{\boldsymbol{\theta}} = \mathbf{t}$ we obtain:

$$\mathbf{J}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}}\dot{\boldsymbol{\theta}} = \dot{\mathbf{t}}$$

Similarly to the previous case we can proceed as follows:

$$\mathbf{J}\ddot{\boldsymbol{\theta}} = \dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\boldsymbol{\theta}}$$

Where:

$$\dot{\mathbf{J}} = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{E}\dot{\boldsymbol{s}}_1 & \boldsymbol{E}\dot{\boldsymbol{s}}_2 & \boldsymbol{E}\dot{\boldsymbol{s}}_3 \end{bmatrix}, \ddot{\boldsymbol{\theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}, \dot{\mathbf{t}} = \begin{bmatrix} \ddot{\phi} \\ \ddot{\mathbf{p}} \end{bmatrix}$$

ACCELERATION ANALYSIS

Considering only two equations (as seen before) we can write:

$$[\mathbf{E}(\boldsymbol{s}_2 - \mathbf{s}_1) \quad \mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)]\begin{bmatrix}\ddot{\theta}_2 \\ \ddot{\theta}_3\end{bmatrix} = \mathbf{w}$$

where **w** is defined as:

$$\mathbf{w} = \ddot{\mathbf{p}} - \mathbf{E}(\dot{\theta}_1\mathbf{s}_1 + \dot{\theta}_2\mathbf{s}_2 + \dot{\theta}_3\mathbf{s}_3 + \ddot{\phi}\mathbf{s}_1)$$

So we can determine the accelerations, which are:

$$\ddot{\theta}_2 = \frac{(\mathbf{s}_3 - \mathbf{s}_1)^T\mathbf{w}}{\Delta}$$

$$\ddot{\theta}_3 = \frac{(\mathbf{s}_2 - \mathbf{s}_1)^T\mathbf{w}}{\Delta}$$

$$\ddot{\theta}_1 = \ddot{\phi} - (\ddot{\theta}_2 + \ddot{\theta}_3)$$

STATIC ANALYSIS

External actions acting on EE of a manipulator can always attributable to a couple *n* and to a force **f**. We can represent them into a single vector

$$\mathbf{w} = \begin{bmatrix} n \\ \mathbf{f} \end{bmatrix}$$

If additionally, we denote by $\boldsymbol{\tau}$ the 3-D vector of joint torques.

$$\mathbf{J}^T \mathbf{w} = \boldsymbol{\tau}$$

Where

$$\mathbf{J}^T = \begin{bmatrix} 1 & (\mathbf{E}\mathbf{s}_1)^T \\ 1 & (\mathbf{E}\mathbf{s}_2)^T \\ 1 & (\mathbf{E}\mathbf{s}_3)^T \end{bmatrix}$$

STATIC ANALYSIS

We subtract the first scalar equation from the second and the third scalar equations, which renders the foregoing system in the form

$$\begin{bmatrix} 1 & (\mathbf{E}\mathbf{s}_1)^T \\ 0 & [\mathbf{E}(\mathbf{s}_2 - s_1)]^T \\ 0 & [\mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)]^T \end{bmatrix} \begin{bmatrix} n \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 - \tau_1 \\ \tau_3 - \tau_1 \end{bmatrix}$$

The last two equations have been decoupled from the first one, which allows us to solve them separately. we have reduced the system to one of two equations in two unknowns, namely

$$\begin{bmatrix} [\mathbf{E}(\mathbf{s}_2 - \mathbf{s}_1)]^T \\ [\mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)]^T \end{bmatrix} \mathbf{f} = \begin{bmatrix} \tau_2 - \tau_1 \\ \tau_3 - \tau_2 \end{bmatrix}$$

STATIC ANALYSIS

From which we readily obtain:

$$\mathbf{f} = \begin{bmatrix} [\mathbf{E}(\mathbf{s}_2 - \mathbf{s}_1)]^T \\ [\mathbf{E}(\mathbf{s}_3 - \mathbf{s}_1)]^T \end{bmatrix}^{-1} \begin{bmatrix} \tau_2 - \tau_1 \\ \tau_3 - \tau_2 \end{bmatrix}$$

and hence, upon expansion of the above inverse

$$\mathbf{f} = \frac{1}{\Delta}[(\tau_2 - \tau_1)(\mathbf{s}_3 - \mathbf{s}_1) - (\tau_3 - \tau_1)(\mathbf{s}_2 - \mathbf{s}_1)]$$

Finally, the resultant moment n acting on the EE is readily calculated from the first scalar equation of the system as:

$$n = \tau_1 + \mathbf{s}_1^T \mathbf{E}\mathbf{f}$$