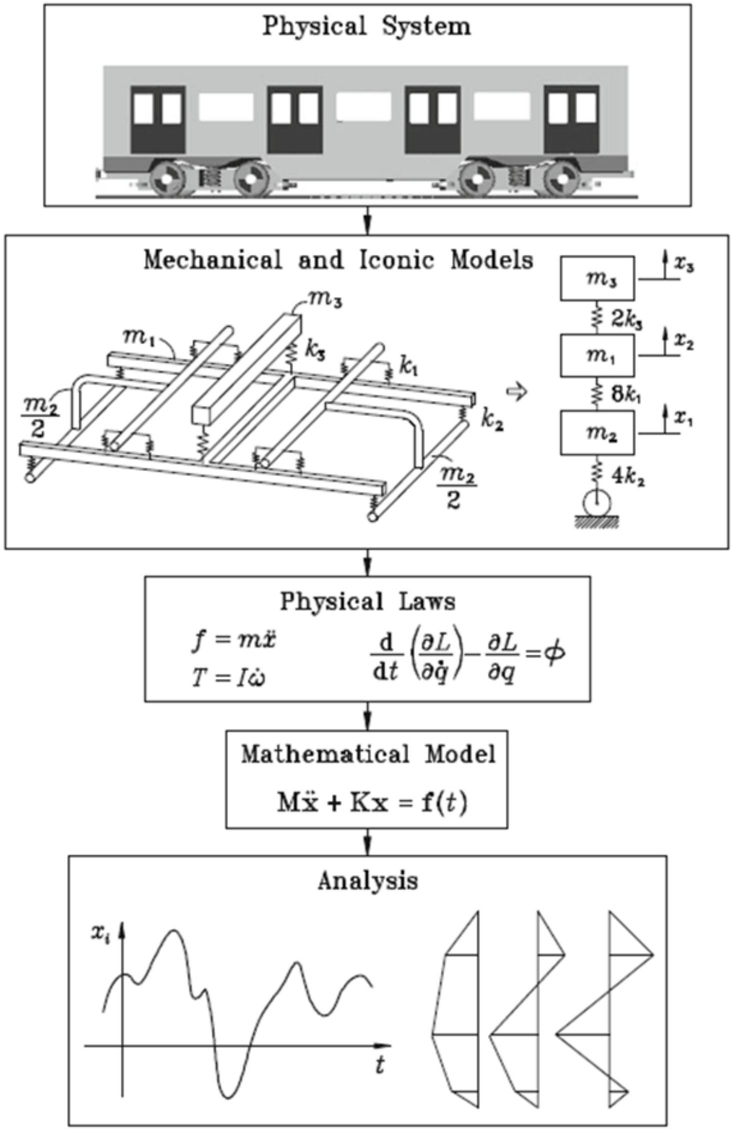
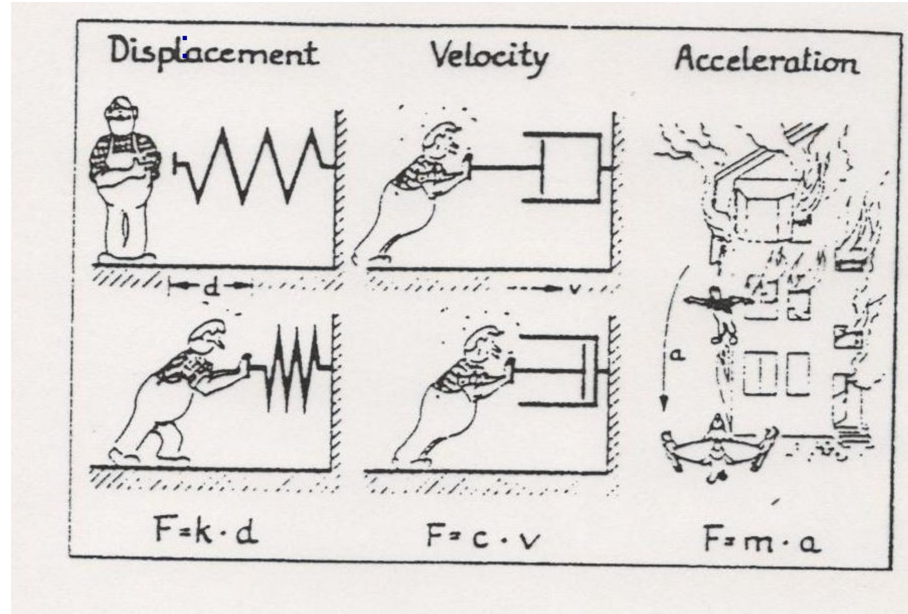
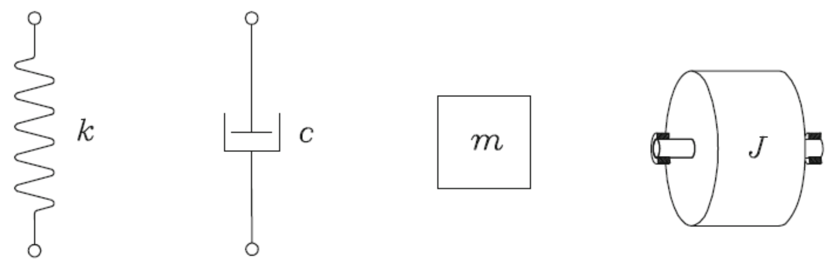
The Modeling of Single/n dof Mechanical Systems







Some lumped-parameter elements

of mechanical systems

- **INGREDIENTS OF A TECHNIQUE MULTIBODY**

  - A SET OF GENERALIZED COORDINATES.

  - A METHOD TO DESCRIBE THE TOPOLOGY OF THE SYSTEM AND THE INTERCONNECTIONS BETWEEN THE BODIES.

  - THE INERTIAL PROPERTIES OF THE MASSES AND A LAW OF MOTION

  - A MATHEMATICAL TOOL TO SOLVE THE EQUATIONS.

technique multibody

1. *Principles of Dynamics*

The Newton-Euler equations:

$$\{\mathcal{F}\}^{(0)} = m\,\{a_G\}^{(0)}$$

$$\{\mathcal{M}_{G_i}\}^{(0)} = \left[J^{(0)}\right]\{\alpha\}^{(0)} + \left[\tilde{\omega}^{(0)}\right]\left[J^{(0)}\right]\{\omega\}^{(0)}$$

Principle of virtual work and the principle of d'Alembert:

$$\delta W = \sum_{k=1}^{N}\left(\vec{F}_k^e - m_k\vec{a}_k\right)\cdot\delta\vec{r}_k = 0$$

Hamilton's principle:

$$\delta\int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2}\delta W_n\,dt = 0$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \quad (j = 1, \ldots, n)$$

Other principles of Dynamics

- Principle of Gauss
- Principle of Gibbs-Jourdain
- Principle Hertz

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2 formalisms of multibody dynamics

The various formalisms used in the generation of the equations of motion can be classified by different criteria:

1. Criterion based on the principles of dynamics used.
   - Eulerian methods
   - Lagrangian methods
   - generalized Lagrangian methods (Gibbs, Jourdain, Kane, etc. ..)

2. Criterion based on the type of the variable used to represent the system's motion
   - generalized coordinates which refer to the absolute motion of the masses, or to an inertial reference frame.
   - generalized coordinates which refer to the relative motion between the bodies interacting.

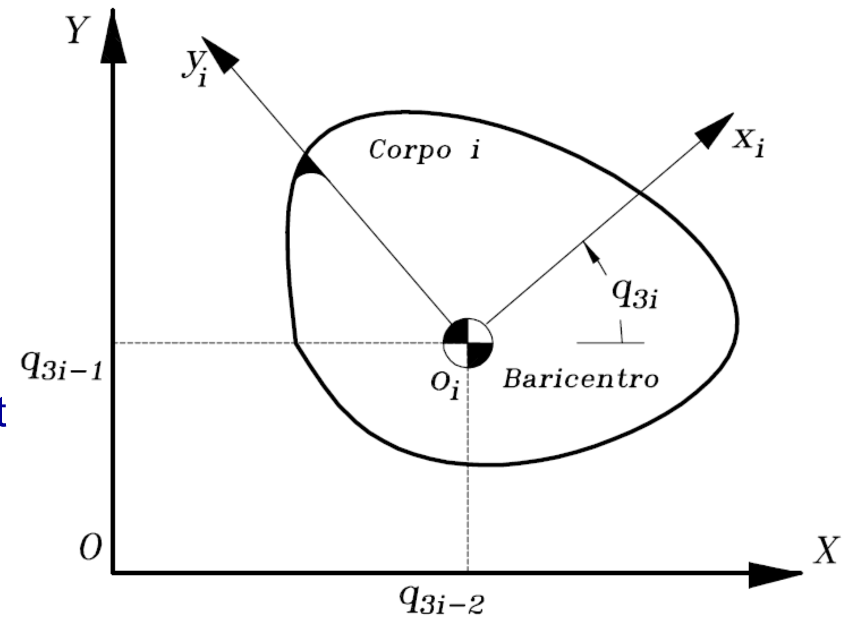
3. Criterion which involves the number of equations used.
   - The set of equations is redundant. You have a lot of equations, but in simple algebraic form. The solution provides not only information on the bike, but also on the forces of constraint.
   - The equations, in number strictly necessary, have a complex algebraic structure and are strongly couple

technique multibody:

3. Position of a body in the plane

$$X = q_{3i-2} + x\cos q_{3i} - y\sin q_{3i}$$
$$Y = q_{3i-1} + x\sin q_{3i} + y\cos q_{3i}$$

- (X, Y) inertial cartesian coordinates of a generic point of the body;
- (x, y) local Cartesian coordinates of the point above;
- (q3i-2, q3i-1, q3i) generalized coordinates of the 'body i.



$$[A_i] = \begin{bmatrix} \cos q_{3i} & -\sin q_{3i} \\ \sin q_{3i} & \cos q_{3i} \end{bmatrix}$$

We have:

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = [A_i] \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} q_{3i-2} \\ q_{3i-1} \end{Bmatrix}$$

technique multibody:

4 Kinematic Constraints

The kinematic constraints can be classified into two categories:

• Structural constraints (scleronomi)
Those due to the presence of the kinematic pairs or particular constraints that do not vary over time. These constraints depend only on the coordinates generalized and the geometry of the system.
- Revolute pair
- Prismatic pair
- Higher torque (e.g. cam-assignor)
- Torque Gear
- Constraints distance

• Constraints of motion (reonomi)
Those due to kinematic laws of motion prescribed to members motives.
In these constraints the time variable t appears explicitly.
Exist many constraints as there are degrees of freedom of motion of the system.

Kinematic analysis: Method of constraint's equations

Analysis of the positions

$$\{\Psi\} \equiv \left\{ \begin{array}{c} \Psi^s \\ \Psi^d \end{array} \right\} = \{0\}$$

vector of structural constraints

vector of the motion constraints

Analysis of the velocity

$$[\Psi_q]\{\dot{q}\} = -\{\Psi_t\}$$

$$[\Psi_q] = \begin{bmatrix} \frac{\partial \Psi_1}{\partial q_1} & \frac{\partial \Psi_1}{\partial q_2} & \cdots & \frac{\partial \Psi_1}{\partial q_n} \\ \frac{\partial \Psi_2}{\partial q_1} & \frac{\partial \Psi_2}{\partial q_2} & \cdots & \frac{\partial \Psi_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi_n}{\partial q_1} & \frac{\partial \Psi_n}{\partial q_2} & \cdots & \frac{\partial \Psi_n}{\partial q_n} \end{bmatrix} \qquad \{\Psi_t\} = \left\{ \begin{array}{c} \frac{d\Psi_1}{dt} \\ \frac{d\Psi_2}{dt} \\ \vdots \\ \frac{d\Psi_n}{dt} \end{array} \right\}$$

Jacobian matrix

Kinematic analysis: Method of constraint's equations

Analysis of the acceleration

$$[\Psi_q]\{\ddot{q}\} = \{\gamma\} \ ,$$

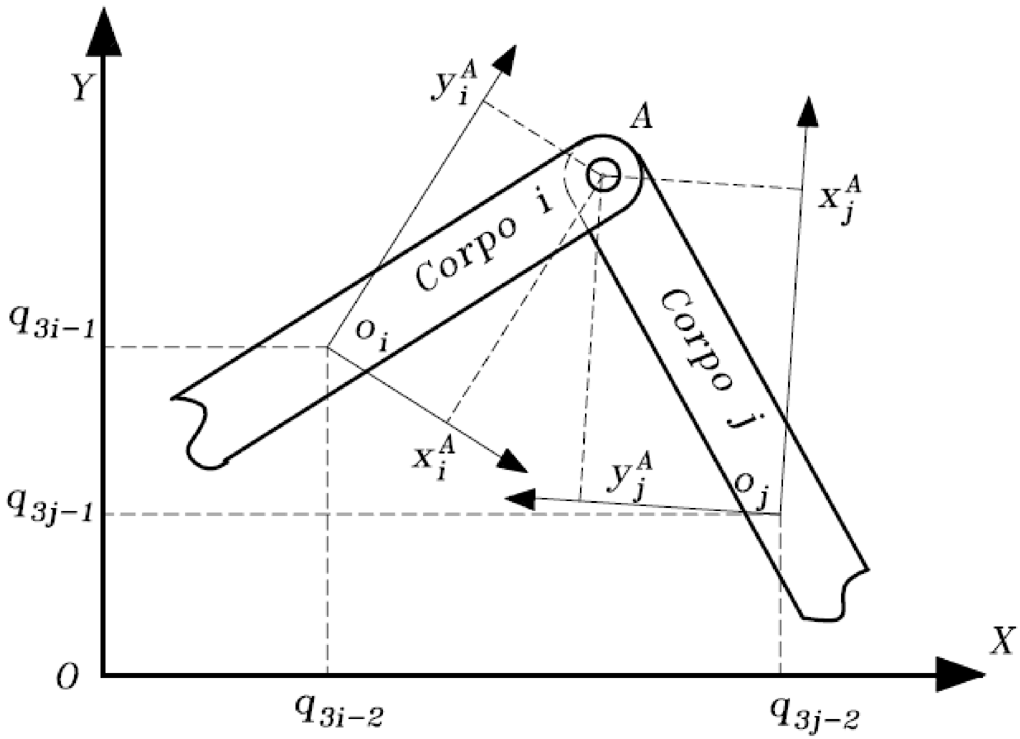
$$\{\gamma\} = -\left([\Psi_q]\{\dot{q}\}\right)_q \{\dot{q}\} - 2[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\}$$

mathematical details:

$$\begin{aligned}
&\frac{d}{dt}\{[\Psi_q(q,t)]\{\dot{q}\}\} \\
= &\left[\frac{d}{dt}\Psi_q(q,t)\right]\{\dot{q}\} + [\Psi_q(q,t)]\{\ddot{q}\} \\
= &\left[\frac{\partial}{\partial q}\{[\Psi_q(q,t)]\{\dot{q}\}\} + \frac{\partial \Psi_q}{\partial t}\right]\{\dot{q}\} + [\Psi_q(q,t)]\{\ddot{q}\}
\end{aligned}$$

$$\frac{d}{dt}\{\Psi_t(q,t)\} = [\Psi_{tq}]\{\dot{q}\} + \{\Psi_{tt}\}$$

Revolute pair



$$X_i^A - X_j^A = 0$$
$$Y_i^A - Y_j^A = 0\ .$$

Revolute pair

$$
\begin{aligned}
\frac{\partial \Psi_k}{\partial q_{3i-2}} &= \Psi_q(k, 3i-2) = 1 \\
\frac{\partial \Psi_k}{\partial q_{3i-1}} &= \Psi_q(k, 3i-1) = 0 \\
\frac{\partial \Psi_k}{\partial q_{3i}} &= \Psi_q(k, 3i) = -\left(x_i^A \sin q_{3i} + y_i^A \cos q_{3i} \right) \\
&= q_{3i-1} - Y_i^A \\
\frac{\partial \Psi_k}{\partial q_{3j-2}} &= \Psi_q(k, 3j-2) = -1 \\
\frac{\partial \Psi_k}{\partial q_{3j-1}} &= \Psi_q(k, 3j-1) = 0 \\
\frac{\partial \Psi_k}{\partial q_{3j}} &= \Psi_q(k, 3j) = x_j^A \sin q_{3j} + y_j^A \cos q_{3j} \\
&= Y_j^A - q_{3j-1}
\end{aligned}
$$

Revolute pair

$$\frac{\partial \Psi_{k+1}}{\partial q_{3i-2}} = \Psi_q(k+1, 3i-2) = 0$$

$$\frac{\partial \Psi_{k+1}}{\partial q_{3i-1}} = \Psi_q(k+1, 3i-1) = 1$$

$$\begin{aligned}\frac{\partial \Psi_{k+1}}{\partial q_{3i}} &= \Psi_q(k+1, 3i) = x_i^A \cos q_{3i} - y_i^A \sin q_{3i} \\ &= X_i^A - q_{3i-2}\end{aligned}$$

$$\frac{\partial \Psi_{k+1}}{\partial q_{3j-2}} = \Psi_q(k+1, 3j-2) = 0$$

$$\frac{\partial \Psi_{k+1}}{\partial q_{3j-1}} = \Psi_q(k+1, 3j-1) = -1$$

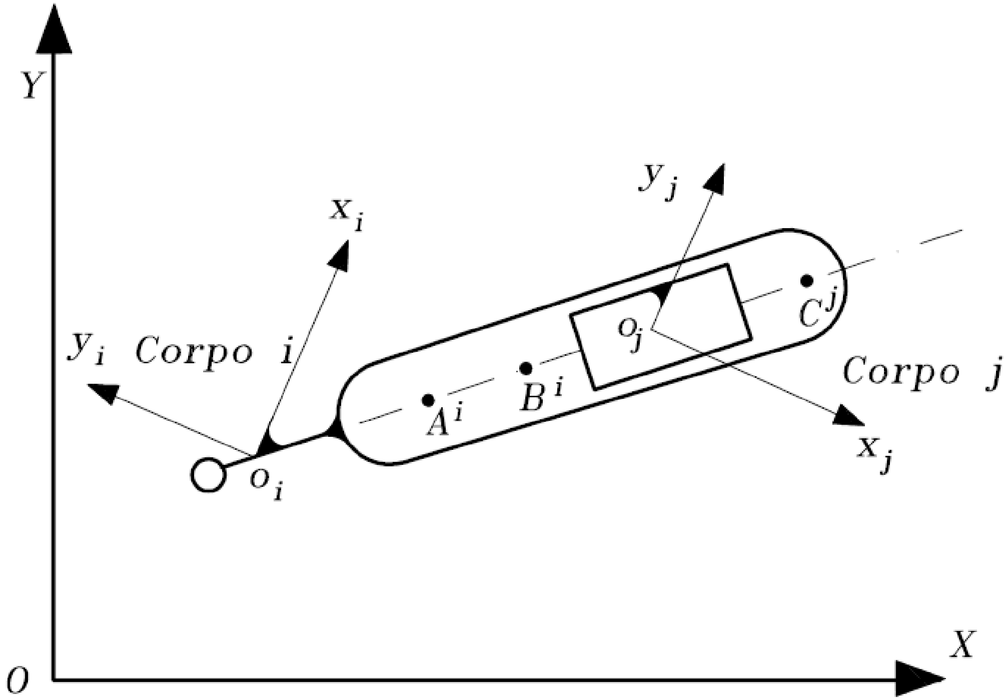
$$\begin{aligned}\frac{\partial \Psi_{k+1}}{\partial q_{3j}} &= \Psi_q(k+1, 3j) = -x_j^A \cos q_{3j} + y_j^A \sin q_{3j} \\ &= q_{3j-2} - X_j^A \ ,\end{aligned}$$

Revolute pair

$$\gamma(k) = -\left[\dot{q}_{3i}^2\left(q_{3i-2} - X_i^A\right) - \dot{q}_{3j}^2\left(q_{3j-2} - X_j^A\right)\right]$$

$$\gamma(k+1) = -\left[\dot{q}_{3i}^2\left(q_{3i-1} - Y_i^A\right) - \dot{q}_{3j}^2\left(q_{3j-1} - Y_j^A\right)\right]$$

Prismatic pair



$$
\Psi_k \equiv \begin{vmatrix} X_i^A & Y_i^A & 1 \\ X_i^B & Y_i^B & 1 \\ X_j^C & Y_j^C & 1 \end{vmatrix} = 0
$$

$$
\Psi_{k+1} \equiv q_{3i} - q_{3j} - \Delta\phi_{ij} = 0
$$

Prismatic pair

$$
\begin{aligned}
\frac{\partial \Psi_k}{\partial q_{3i-2}} &= \Psi_q(k, 3i-2) = Y_i^B - Y_i^A \\
\frac{\partial \Psi_k}{\partial q_{3i-1}} &= \Psi_q(k, 3i-1) = X_i^A - X_i^B \\
\frac{\partial \Psi_k}{\partial q_{3i}} &= \Psi_q(k, 3i) = -\left(X_i^B - X_i^A\right)\left(X_j^C - q_{3i-2}\right) \\
&\quad - \left(Y_i^B - Y_i^A\right)\left(Y_j^C - q_{3i-1}\right) \\
\frac{\partial \Psi_k}{\partial q_{3j-2}} &= \Psi_q(k, 3j-2) = Y_i^A - Y_i^B \\
\frac{\partial \Psi_k}{\partial q_{3j-1}} &= \Psi_q(k, 3j-1) = X_i^B - X_i^A \\
\frac{\partial \Psi_k}{\partial q_{3j}} &= \Psi_q(k, 3j) = -\left(X_i^B - X_i^A\right)\left(q_{3j-2} - X_j^C\right) \\
&\quad - \left(Y_i^A - Y_i^B\right)\left(Y_j^C - q_{3j-1}\right)
\end{aligned}
$$

Prismatic pair

$$\frac{\partial \Psi_k + 1}{\partial q_{3i-2}} = \Psi_q(k+1, 3i-2) = 0$$

$$\frac{\partial \Psi_k + 1}{\partial q_{3i-1}} = \Psi_q(k+1, 3i-1 = 0$$

$$\frac{\partial \Psi_k + 1}{\partial q_{3i}} = \Psi_q(k+1, 3i) = 1$$

$$\frac{\partial \Psi_k + 1}{\partial q_{3j-2}} = \Psi_q(k+1, 3j-2) = 0$$

$$\frac{\partial \Psi_k + 1}{\partial q_{3j-1}} = \Psi_q(k+1, 3j-1) = 0$$

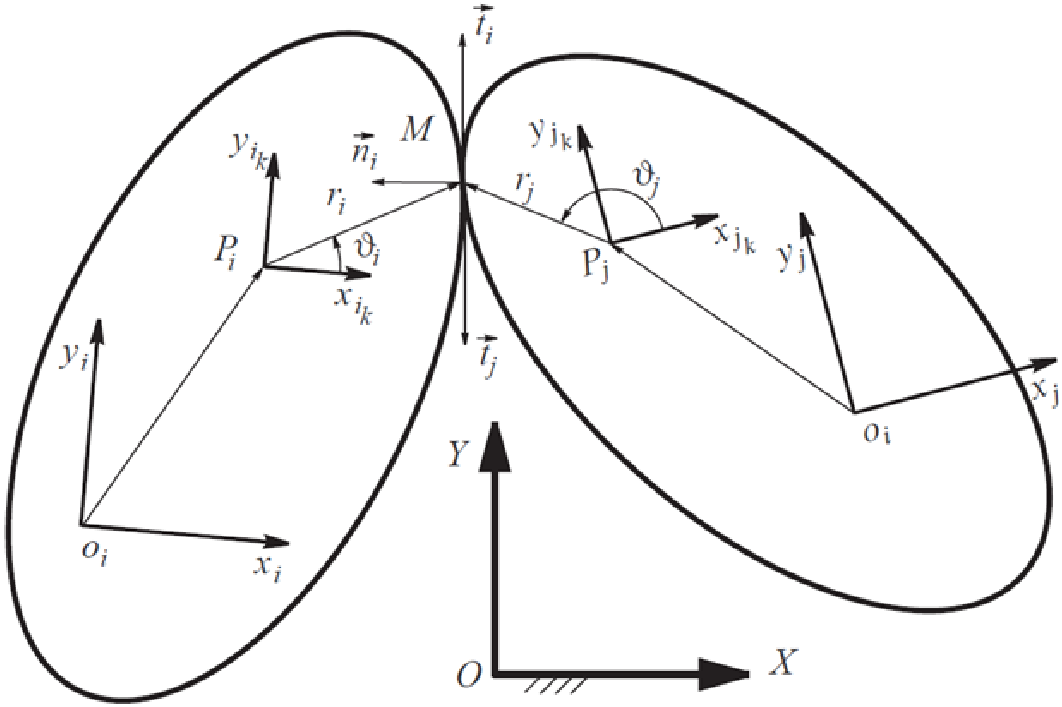
$$\frac{\partial \Psi_k + 1}{\partial q_{3j}} = \Psi_q(k+1, 3j) = -1\ .$$

Prismatic pair

$$
\begin{aligned}
\gamma(k) = 2\dot{q}_{3i} \Big[&\left(Y_i^A - Y_i^B\right) \left(\dot{q}_{3i-1} - \dot{q}_{3j-1}\right) \\
&+ \left(X_i^A - X_i^B\right) \left(\dot{q}_{3i-2} - \dot{q}_{3j-2}\right) \Big] \\
&+ \dot{q}_{3i}^2 \Big[\left(X_i^A - X_i^B\right) \left(q_{3i-1} - q_{3j-1}\right) \\
&+ \left(Y_i^B - Y_i^A\right) \left(q_{3i-2} - q_{3j-2}\right) \Big] \\
\gamma(k+1) = 0 \ .
\end{aligned}
$$

UPPER KINEMATIC PAIRS

With reference to Figure 4, introduced a system of axes Cartesian Pi - xikyik oriented like that or - xiyi, the coordinates point M of the generic profile Ci, belonging to the body i, are given by parametric equations:



$$x_{i_k}^M = r_i\left(\vartheta_i\right)\cos\vartheta_i$$
$$y_{i_k}^M = r_i\left(\vartheta_i\right)\sin\vartheta_i$$

UPPER KINEMATIC PAIRS

Coincidence between the Cartesian coordinates of the point M, considered as belonging both the body and the body j:

$$X_i^M - X_j^M = 0$$

$$Y_i^M - Y_j^M = 0$$

Orthogonality of {ni} and {tj}:

$$\{n_i\}^T \{t_j\} = 0$$

if

$$r_i' = \frac{\partial r_i}{\partial \vartheta_i} \; ,$$

the tangent to Ci in M appears to be parallel to the vector

$$\{t_i\}^{(i)} = \left\{ \begin{array}{c} \dfrac{\partial x_{i_k}^M}{\partial \vartheta_i} \\ \dfrac{\partial y_{i_k}^M}{\partial \vartheta_i} \end{array} \right\} = \left\{ \begin{array}{c} r_i' \cos \vartheta_i - r_i \sin \vartheta_i \\ r_i' \sin \vartheta_i + r_i \cos \vartheta_i \end{array} \right\}$$

UPPER KINEMATIC PAIRS

The normal ni to Ci in M is parallel to the vector obtained from the product

$$\{n_i\}^{(i)} = [R]\,\{t_i\}^{(i)}$$

where

$$[R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The following transformations provide the components in O-XY tangent and normal vectors to Ci in M

$$\{t_i\} = [A_i]\,\{t_i\}^{(i)}\,,$$

$$\{n_i\} = [A_i]\,\{n_i\}^{(i)}\,.$$

$$\left[\begin{array}{cc|c} \frac{\partial \Psi_k}{\partial q_{3i-2}} & \frac{\partial \Psi_k}{\partial q_{3i-1}} & \frac{\partial \Psi_k}{\partial q_{3i}} \\ \frac{\partial \Psi_{k+1}}{\partial q_{3i-2}} & \frac{\partial \Psi_{k+1}}{\partial q_{3i-1}} & \frac{\partial \Psi_{k+1}}{\partial q_{3i}} \\ \hline \frac{\partial \Psi_{k+2}}{\partial q_{3i-2}} & \frac{\partial \Psi_{k+2}}{\partial q_{3i-1}} & \frac{\partial \Psi_{k+2}}{\partial q_{3i}} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$[B_i]\begin{Bmatrix} x_i^M \\ y_i^M \end{Bmatrix}$} \\ 0 & 1 & \\ \hline 0 & 0 & -\{t_i\}^{(i)T}[A_{ij}]\{t_j\}^{(j)} \end{array}\right],$$

$$\left[\begin{array}{cc|c} \frac{\partial \Psi_k}{\partial q_{3j-2}} & \frac{\partial \Psi_k}{\partial q_{3j-1}} & \frac{\partial \Psi_k}{\partial q_{3j}} \\ \frac{\partial \Psi_{k+1}}{\partial q_{3j-2}} & \frac{\partial \Psi_{k+1}}{\partial q_{3j-1}} & \frac{\partial \Psi_{k+1}}{\partial q_{3j}} \\ \hline \frac{\partial \Psi_{k+2}}{\partial q_{3j-2}} & \frac{\partial \Psi_{k+2}}{\partial q_{3j-1}} & \frac{\partial \Psi_{k+2}}{\partial q_{3j}} \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & \multirow{2}{*}{$-[B_j]\begin{Bmatrix} x_j^M \\ y_j^M \end{Bmatrix}$} \\ 0 & -1 & \\ \hline 0 & 0 & \{t_i\}^{(i)T}[A_{ij}]\{t_j\}^{(j)} \end{array}\right],$$

$$\left[\begin{array}{c|c} \frac{\frac{\partial \Psi_k}{\partial \vartheta_i}}{\frac{\partial \Psi_{k+1}}{\partial \vartheta_i}} & \frac{\frac{\partial \Psi_k}{\partial \vartheta_j}}{\frac{\partial \Psi_{k+1}}{\partial \vartheta_j}} \\ \hline \frac{\partial \Psi_{k+2}}{\partial \vartheta_i} & \frac{\partial \Psi_{k+2}}{\partial \vartheta_j} \end{array}\right] = \left[\begin{array}{c|c} [A_i]\{t_i\}^{(i)} & -[A_j]\{t_j\}^{(j)} \\ \hline -\{t_i\}_{\vartheta_i}^{(i)T}[B_{ij}]\{t_j\}^{(j)} & -\{t_i\}^{(i)T}[B_{ij}]\{t_j\}_{\vartheta_j}^{(j)} \end{array}\right].$$

UPPER KINEMATIC PAIRS

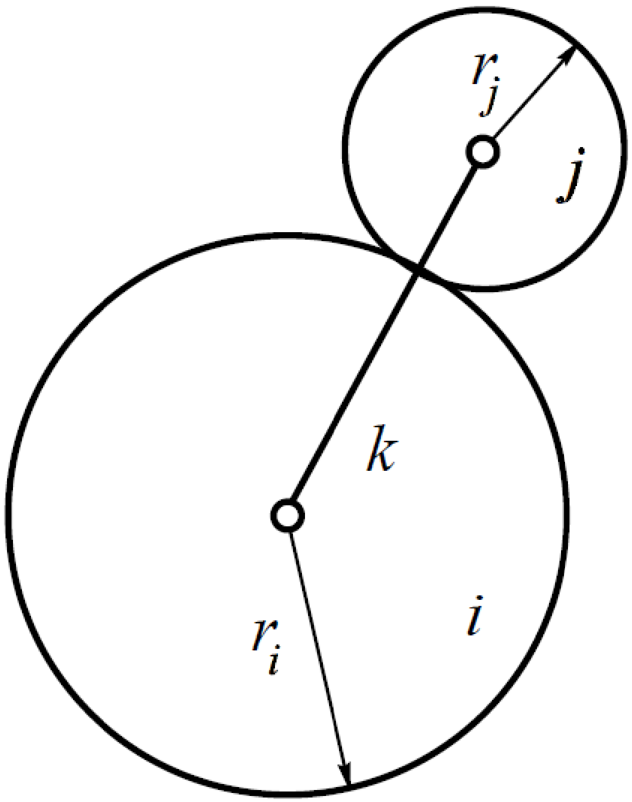
$$\left\{\begin{array}{c}\gamma(k)\\ \gamma(k+1)\end{array}\right\} = \dot{q}_{3i}^2\left[A_i\right]\left\{\begin{array}{c}x_i^M\\ y_i^M\end{array}\right\} - \dot{q}_{3j}^2\left[A_j\right]\left\{\begin{array}{c}x_j^M\\ y_j^M\end{array}\right\}$$
$$+\dot{\vartheta}_j^2\left[A_j\right]\left\{t_j\right\}_{\vartheta_j}^{(j)} - \dot{\vartheta}_i^2\left[A_i\right]\left\{t_i\right\}_{\vartheta_i}^{(i)}$$
$$-2\dot{q}_{3i}\dot{\vartheta}_i\left[B_i\right]\left\{t_i\right\}^{(i)} + 2\dot{q}_{3j}\dot{\vartheta}_j\left[B_j\right]\left\{t_j\right\}^{(j)}$$

$$\gamma(k+2) = -2\left(\dot{q}_{3j}-\dot{q}_{3i}\right)\left(\dot{\vartheta}_i\left\{t_i\right\}_{\vartheta_i}^{(i)T}\left[A_{ij}\right]\left\{t_j\right\}^{(j)} + \dot{\vartheta}_j\left\{t_i\right\}^{(i)T}\left[A_{ij}\right]\left\{t_j\right\}_{\vartheta_j}^{(j)}\right)$$
$$+\dot{\vartheta}_i^2\left\{t_i\right\}_{\vartheta_i\vartheta_i}^{(i)T}\left[B_{ij}\right]\left\{t_j\right\}^{(j)} + \dot{\vartheta}_j^2\left\{t_i\right\}^{(i)T}\left[B_{ij}\right]\left\{t_j\right\}_{\vartheta_j\vartheta_j}^{(j)}$$
$$+2\dot{\vartheta}_i\dot{\vartheta}_j\left\{t_i\right\}_{\vartheta_i}^{(i)T}\left[B_{ij}\right]\left\{t_j\right\}_{\vartheta_j}^{(j)} \quad ,$$

where

$$\left\{t_i\right\}_{\vartheta_i\vartheta_i}^{(i)} = \frac{\partial^2\left\{t_i\right\}^{(i)}}{\partial\vartheta_i^2} = \left\{\begin{array}{c} r_i''' \cos\vartheta_i - 3r_i'' \sin\vartheta_i - 3r_i' \cos\vartheta_i + r_i \sin\vartheta_i \\ r_i''' \sin\vartheta_i + 3r_i'' \cos\vartheta_i - 3r_i' \sin\vartheta_i - r_i \cos\vartheta_i \end{array}\right\}$$

Pair with gears



$$\omega_{ik} r_i = \omega_{jk} r_j$$

Substituting the relations:

$$\omega_{ik} = \dot{q}_{3i} - \dot{q}_{3k}$$

$$\omega_{jk} = \dot{q}_{3j} - \dot{q}_{3k}$$

is obtained

$$\frac{\dot{q}_{3i} - \dot{q}_{3k}}{\dot{q}_{3j} - \dot{q}_{3k}} = \frac{r_j}{r_i} \ ,$$

where $\tau = \dfrac{r_j}{r_i}\ , \quad \tau = \pm \dfrac{z_j}{z_i}\ ,$

Polar of the relative motion between the members i and j

Pair with gears

- Then we have:

$$\dot{q}_{3i} - \tau \dot{q}_{3j} + (\tau - 1)\, \dot{q}_{3k} = 0$$

- For integration of the latter will obtain the constraint equation

$$\Psi \equiv q_{3i} - q^0_{3i} - \tau \left(q_{3j} - q^0_{3j} \right) + (\tau - 1) \left(q_{3k} - q^0_{3k} \right) = 0$$

- where q03i, q03j, q03k are the initial angular positions of the members i, j and k, respectively.

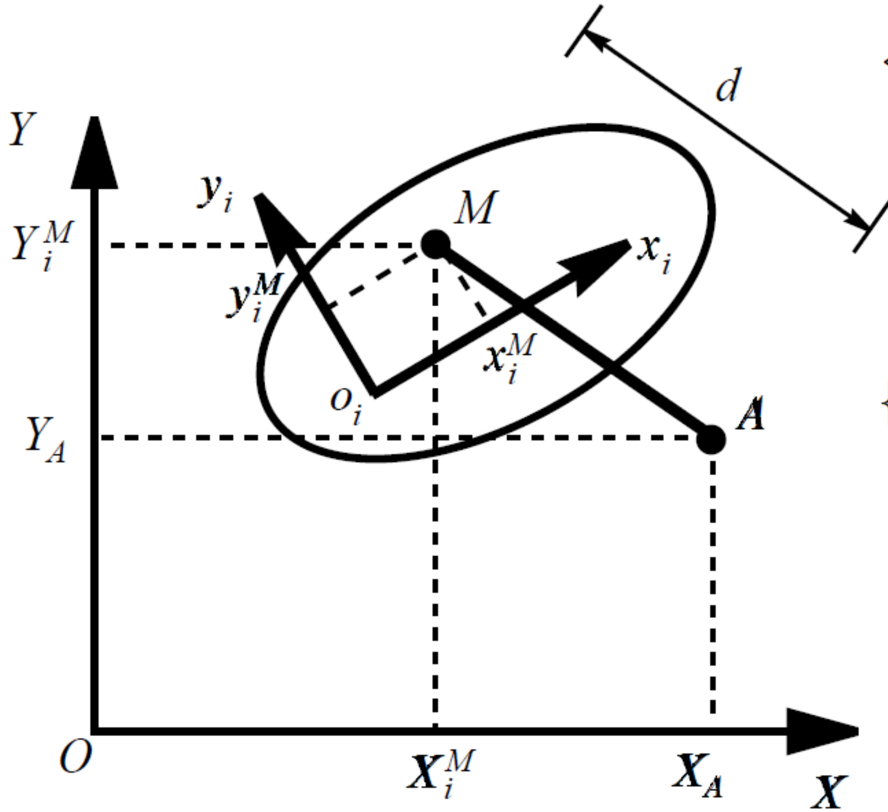
Pair with gears

The elements of the Jacobian matrix associated to this constraint are:

$$
\begin{aligned}
&\frac{\partial \Psi}{\partial q_{3i-2}} = 0, \quad \frac{\partial \Psi}{\partial q_{3i-1}} = 0, \frac{\partial \Psi}{\partial q_{3i}} = 1 \\
&\frac{\partial \Psi}{\partial q_{3j-2}} = 0, \quad \frac{\partial \Psi}{\partial q_{3j-1}} = 0, \frac{\partial \Psi}{\partial q_{3j}} = -\tau \\
&\frac{\partial \Psi}{\partial q_{3k-2}} = 0, \quad \frac{\partial \Psi}{\partial q_{3k-1}} = 0, \frac{\partial \Psi}{\partial q_{3k}} = \tau - 1
\end{aligned}
$$

The element of the vector γ relative to this constraint is always null.

Constraint on distance



$$\left\{ \begin{array}{c} X_i^M \\ Y_i^M \end{array} \right\} = \left\{ \begin{array}{c} q_{3i-2} \\ q_{3i-1} \end{array} \right\} + [A_i] \{s_i^M\}$$

$$\{D\} = \left\{ \begin{array}{c} D_x \\ D_y \end{array} \right\} = \left\{ \begin{array}{c} X_i^M - X^A \\ Y_i^M - Y^A \end{array} \right\}$$

$$\Psi^{ad} \equiv \{D\}^T \{D\} - d^2 = 0 \ .$$

Constraint on distance

- Row of the Jacobian matrix

$$\frac{\partial \Psi^{ad}}{\partial q_{3i-2}} = 2D_x \ ,$$
$$\frac{\partial \Psi^{ad}}{\partial q_{3i-1}} = 2D_y \ ,$$
$$\frac{\partial \Psi^{ad}}{\partial q_{3i}} = 2\left\{s_i^M\right\}^T [B_i]^T \{D\}$$

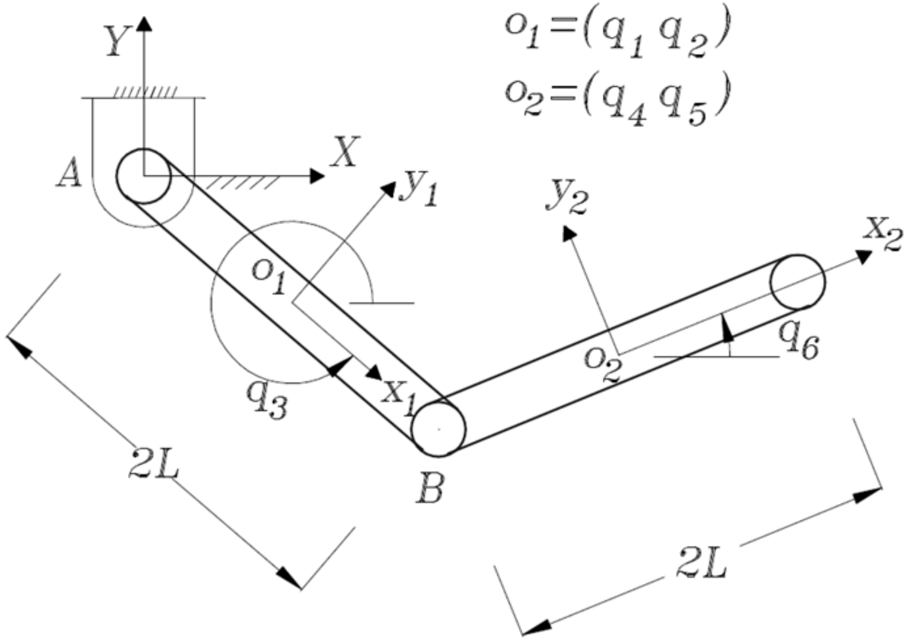
The element of the vector γ relative to this constraint is:

$$\begin{aligned}\left([\Psi_{q_i}]\{\dot{q}_i\}\right)_{q_i}\{\dot{q}_i\} =& 2\Big(\{\dot{r}_i\}^T\{\dot{r}_i\} + 2\left\{s_i^M\right\}^T[B_i]^T\{\dot{r}_i\}\dot{q}_{3i} \\ & -\left\{s_i^M\right\}^T[A_i]^T\{D\}\dot{q}_{3i}^2 + \left\{s_i^M\right\}^T[B_i]^T[B_i]\left\{s_i^M\right\}\dot{q}_{3i}^2\Big) \\ =& -2\left(\left\{\dot{D}\right\}^T\left\{\dot{D}\right\} - \left\{s_i^M\right\}^T[A_i]^T\{D\}\dot{q}_{3i}^2\right) \ . \qquad (31d)\end{aligned}$$

Other constraints on the distances

- Constraint variable distance between two points;
- Constraints on the difference between the ordinates;
- Constraints on the difference between the x-axis;
- Constraints on the difference between angular positions;

Applications: robotic manipulator to 2 g.d.l.



$$q_3 = h(t)$$
$$q_6 = k(t)$$

$$\{\Psi^s\} \equiv \left\{ \begin{array}{c} q_1 - L\cos q_3 \\ q_2 - L\sin q_3 \\ q_1 + L\cos q_3 - q_4 + L\cos q_6 \\ q_2 + L\sin q_3 - q_5 + L\sin q_6 \end{array} \right\} = \{0\}$$

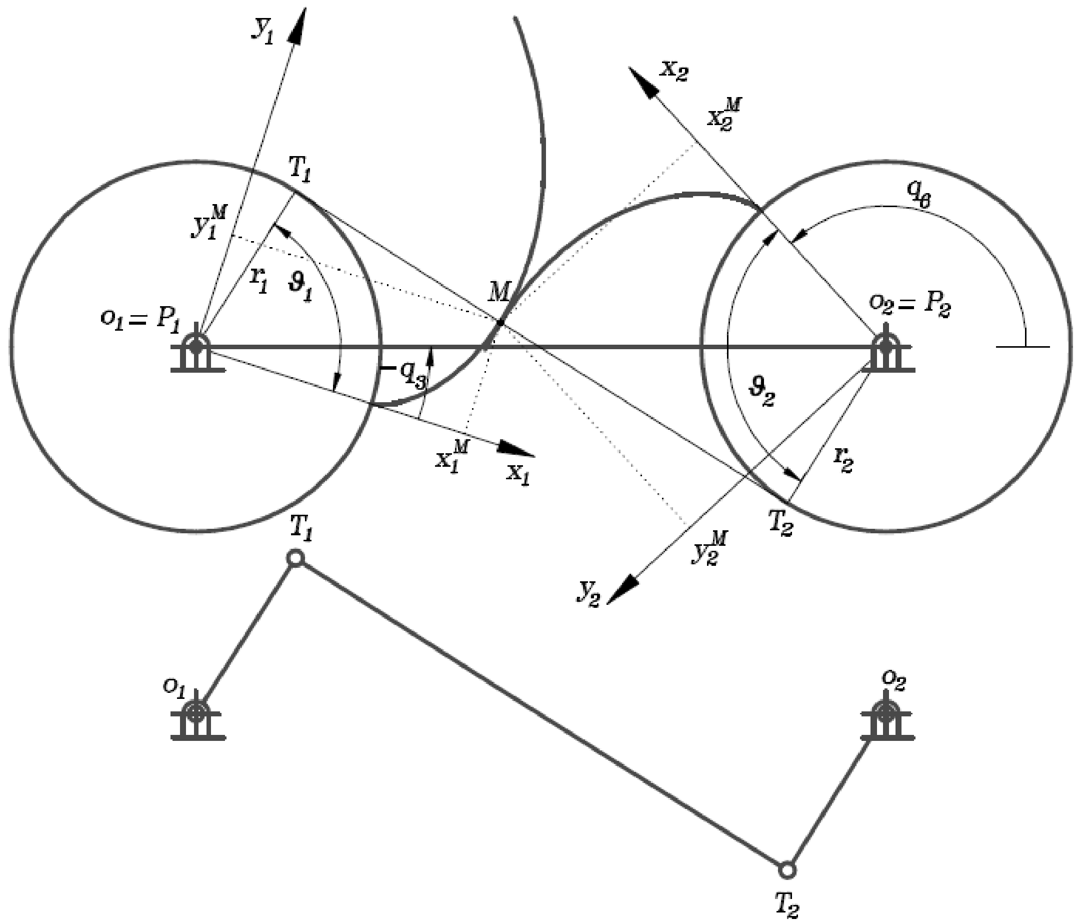
$$\{\Psi^d\} \equiv \left\{ \begin{array}{c} q_3 - h(t)\ , \\ q_6 - k(t)\ . \end{array} \right\} = \{0\}$$

Applications: robotic manipulator to 2 g.d.l.

$$[\Psi_q] = \begin{bmatrix} 1 & 0 & L\sin q_3 & 0 & 0 & 0 \\ 0 & 1 & -L\cos q_3 & 0 & 0 & 0 \\ 1 & 0 & -L\sin q_3 & -1 & 0 & -L\sin q_6 \\ 0 & 1 & L\cos q_3 & 0 & -1 & L\cos q_6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \{\Psi_t\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dot{h}(t) \\ \dot{k}(t) \end{Bmatrix}$$

$$\{\gamma\} = \begin{Bmatrix} -\dot{q}_3^2 L\cos q_3 \\ -\dot{q}_3^2 L\sin q_3 \\ \dot{q}_3^2 L\cos q_3 + \dot{q}_6^2 L\cos q_6 \\ \dot{q}_3^2 L\sin q_3 + \dot{q}_6^2 L\sin q_6 \\ -\ddot{h}(t) \\ -\ddot{k}(t) \end{Bmatrix}$$

Example more…



Summary kinematic analysis

- The kinematic analysis is reduced to the solution of the following systems of equations:

$$\{\Psi\} = \{0\}$$
$$[\Psi_q]\{\dot{q}\} = \{\Psi_t\}$$
$$[\Psi_q]\{\ddot{q}\} = \{\gamma\}$$

- For various constraints were deducted constraint equations for various pairs

kinematics, the elements of the Jacobian matrix and the γ.

Formulation of the equations of dynamics

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \delta W_n dt = 0$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \quad (j = 1, \ldots, n)$$

where

$$L = T - V$$

Minimization of constraintsfunction

Having used a number of coordinates higher than that of dof, the Lagrangian function

$$L = T - V$$

will be subject to the constraint equations of the generalized coordinates

$$\{\Psi\} = \{0\}$$

Method of Lagrange multipliers

$$L^* = L - \lambda\Psi$$

For *L*(q1, q2) subject to PSI(q1, q2) = 0

From mathematical analysis, the conditions of stationarity to turn out to be:

$$\frac{\partial L^*}{\partial q_1} \equiv \frac{\partial L}{\partial q_1} - \lambda\frac{\partial \Psi}{\partial q_1} = 0$$
$$\frac{\partial L^*}{\partial q_2} \equiv \frac{\partial L}{\partial q_2} - \lambda\frac{\partial \Psi}{\partial q_2} = 0$$
$$\frac{\partial L^*}{\partial \lambda} \equiv \Psi = 0 \ .$$

Method of Lagrange multipliers

EXAMPLE: a linear system of equations in which [A] is a rectangular matrix with fewer rows than columns:

$$[A]\{x\} = \{b\}$$

minimum Euclidean norm: $\min \{x\}^T \{x\}$

$$L = \{x\}^T \{x\} - \{\lambda\}^T ([A]\{x\} - \{b\}) \quad ,$$

Applying the condition of stazionarity.

$$\frac{\partial L}{\partial \{x\}} = 2\{x\} - [A]^T \{\lambda\} = \{0\}$$

We have:

$$\{x\} = \frac{1}{2} [A]^T \{\lambda\} \quad .$$

Method of Lagrange multipliers

Moreover:

$$\frac{\partial L}{\partial \{\lambda\}} = [A]\{x\} - \{b\} = \{0\}$$

$$\{\lambda\} = 2\left([A][A]^T\right)^{-1}\{b\},$$

the algebraic solution of constrained optimization problem

$$\{x\} = [A]^T\left([A][A]^T\right)^{-1}\{b\},$$

Extended Lagrangian

$$L = T - V - (\lambda_1 \Psi_1 + \ldots + \lambda_p \Psi_p) \ ,$$

Applying the Lagrange equations;

$$[M]\{\ddot{q}\} + \left[\Psi_q^T\right]\{\lambda\} = \{F_e\}$$

[M] is the mass matrix;

{Fe} is the generalized vector of force;

$$F_j = \frac{\delta W}{\delta q_j} = \sum_{k=1}^{N} \vec{F}_k^{nc} \cdot \frac{\partial \vec{r}_k}{\partial q_j} \ ,$$

- [PSIq]T is the transposed Jacobian matrix associated with the system of $\{\Psi\} = \{0\}$ equations

Extended Lagrangian

Complete system

$$[M]\{\ddot{q}\} + \left[\Psi_q^T\right]\{\lambda\} = \{F_e\}$$

$$\{\Psi\} = 0$$

Or

$$\left\{\ddot{\Psi}\right\} \equiv [\Psi_q]\{\ddot{q}\} - \{\gamma\} = \{0\}$$

$$\begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix} \left\{ \begin{array}{c} \ddot{q} \\ \lambda \end{array} \right\} = \left\{ \begin{array}{c} F_e \\ \gamma \end{array} \right\},$$

Extended Lagrangian

Numerical solutions

$$\{\dot{q}\}_{(i+1)\triangle t} = \{\dot{q}\}_{i\triangle t} + \{\ddot{q}\}_{i\triangle t}\,\triangle t \ ,$$

$$\{q\}_{(i+1)\triangle t} = \{q\}_{i\triangle t} + \{\dot{q}\}_{i\triangle t}\,\triangle t + \frac{1}{2}\,\{\ddot{q}\}_{i\triangle t}\,\triangle t^2 \ .$$

Computational artifice

$$\begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$[C_{22}] = \left([\Psi_q] \, [M]^{-1} \, [\Psi_q]^T \right)^{-1} \quad ,$$

$$[C_{11}] = [M]^{-1} - [M]^{-1} \, [\Psi_q]^T \, [C_{22}] \, [\Psi_q] \, [M]^{-1} \quad ,$$

$$[C_{12}] = [C_{21}]^T = - [M]^{-1} \, [\Psi_q]^T \, [C_{22}] \quad .$$

Solution:

$$\{\ddot{q}\} = [C_{11}] \, \{F_e\} + [C_{12}] \, \{\gamma\} \quad ,$$

$$\{\lambda\} = [C_{21}] \, \{F_e\} + [C_{22}] \, \{\gamma\} \quad .$$

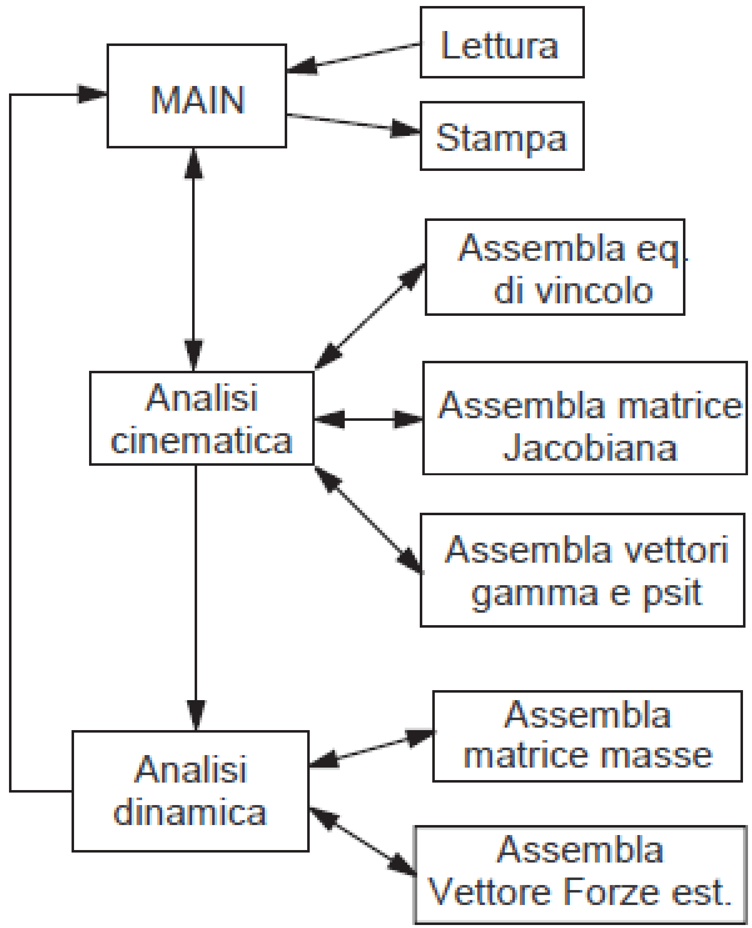
Inverse dynamics

For inverse dynamics analysis refers to the calculation of the driving forces and those of constraint compatible with a prescribed kinematic state of the mechanical system.

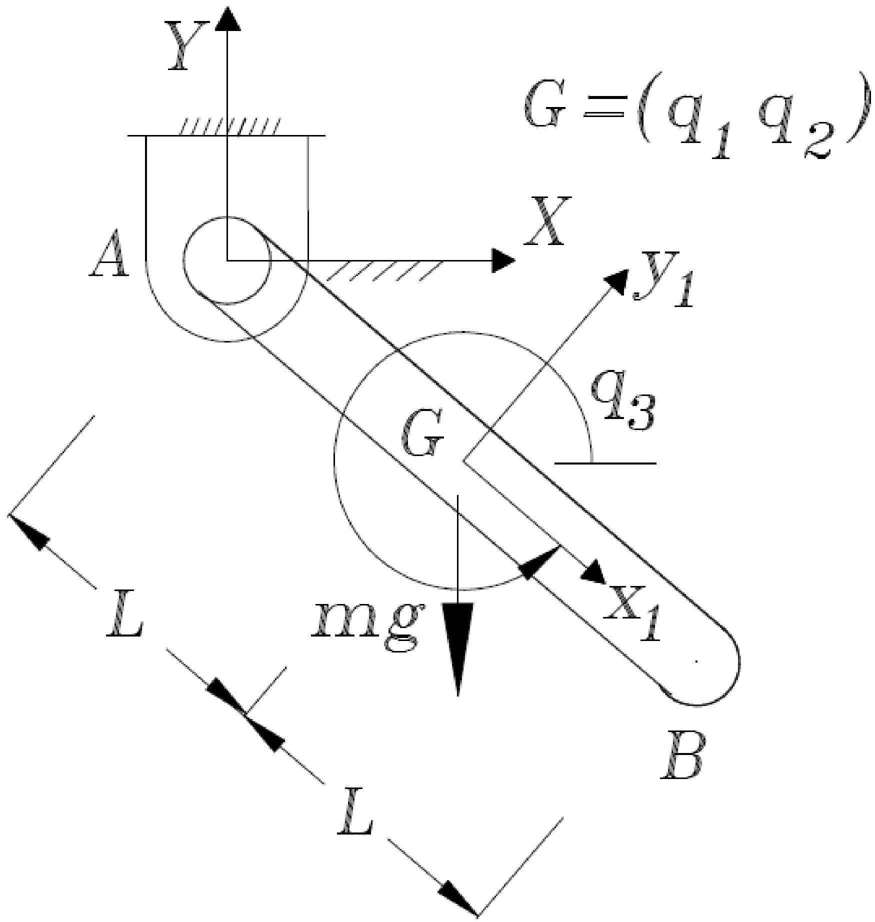
$$[\Psi_q]^T \{\lambda\} = -[M]\{\ddot{q}\} + \{F_e\} + \{Q_a(q, \dot{q}, \lambda)\}$$

- {q}: the vector of the generalized coordinates (3 for each body)
- [PSIq]: is the Jacobian matrix of the constraints (scleronomi and reonomi);
- {lamda} is the vector of Lagrange multipliers associated with the above-mentioned constraints;
- {Fe}: is the vector of external forces;
- {Qa(q, q˙,lamda)}: is the vector of possible friction forces in the revolute pairs, prismatic and gear.

Flow-chart of the analysis of inverse dynamics

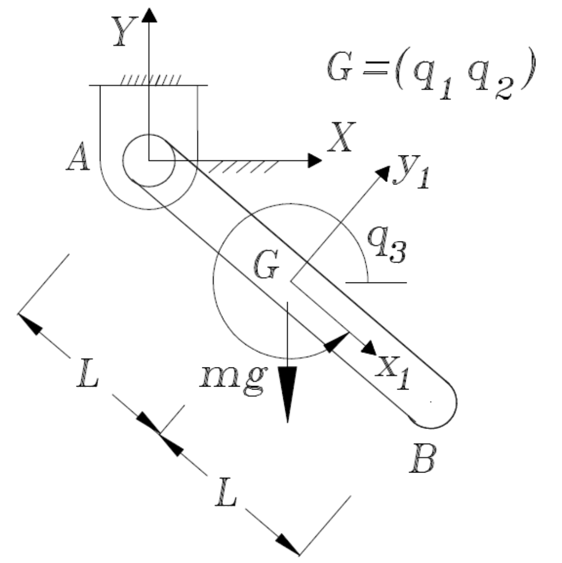


Applications: Simple pendulum



$$\left(I_G + mL^2\right)\ddot{\vartheta} + mgL\sin\vartheta = \mathcal{T} \ ,$$

Applications: Simple pendulum



$$\Psi_1 \equiv q_1 - x_G \cos q_3 = 0$$

$$\Psi_2 \equiv q_2 - x_G \sin q_3 = 0$$

$$L = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_G \dot{q}_3^2 - \lambda_1 \Psi_1 - \lambda_2 \Psi_2$$

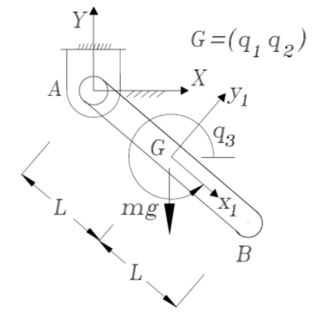
$$\delta W = \mathcal{T} \delta q_3 - mg\delta q_2$$

$$m\ddot{q}_1 + \lambda_1 = 0$$

$$m\ddot{q}_2 + \lambda_2 + mg = 0$$

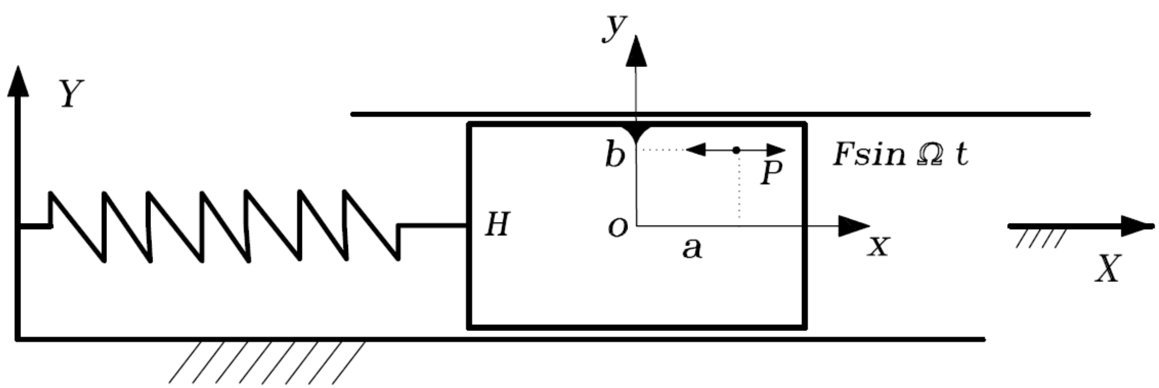
$$I_G \ddot{q}_3 + \lambda_1 x_G \sin q_3 - \lambda_2 x_G \cos q_3 = \mathcal{T}$$

Applications: Simple pendulum



$$
\begin{bmatrix}
m & 0 & 0 & 1 & 0 \\
0 & m & 0 & 0 & 1 \\
0 & 0 & I_G & x_G \sin q_3 & -x_G \cos q_3 \\
1 & 0 & x_G \sin q_3 & 0 & 0 \\
0 & 1 & -x_G \cos q_3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \lambda_1 \\ \lambda_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\ -mg \\ \mathcal{T} \\ -x_G \dot{q}_3^2 \cos q_3 \\ -x_G \dot{q}_3^2 \sin q_3
\end{bmatrix}
$$

Mass-spring system



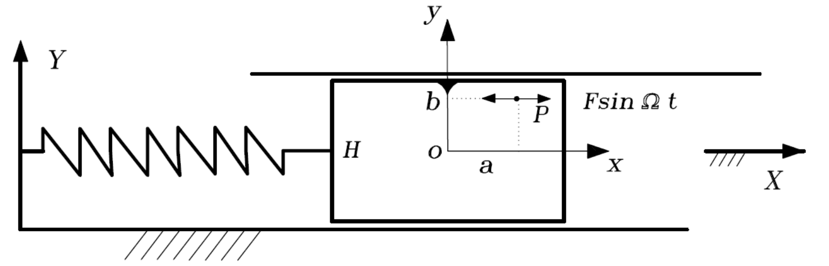
$$\{F\} = \left\{ \begin{array}{c} F \sin \Omega t \\ 0 \\ -bF \sin \Omega t \end{array} \right\}$$

$$\Psi_1 \equiv \left| \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ q_1 & q_2 & 1 \end{array} \right| = q_2 = 0,$$

$$\Psi_2 \equiv q_3 = 0.$$

$$[\Psi_q] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

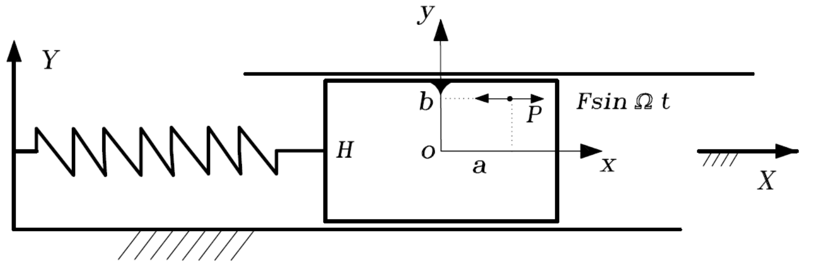
Mass-spring system



$$L = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{I}{2}\dot{q}_3^2 - \frac{K}{2}\left(q_1 - l_0\right)^2 - \lambda_1\Psi_1 - \lambda_2\Psi_2$$

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} F\sin\Omega t - K\left(q_1 - l_0\right) \\ 0 \\ -bF\sin\Omega t \\ 0 \\ 0 \end{Bmatrix}.$$

Mass matrix

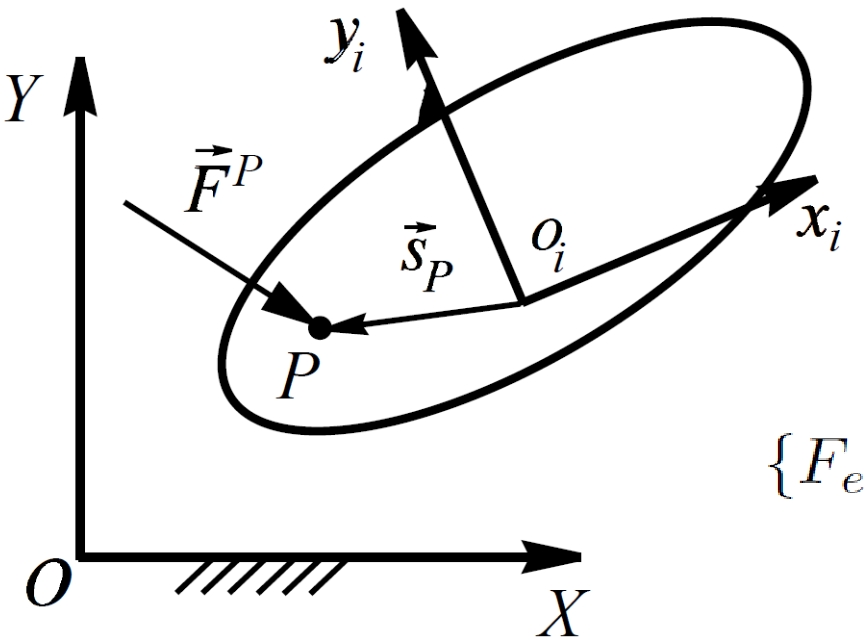


$$[M] = \left[\begin{array}{c:c:c:c} M_1 & 0 & 0 & 0 \\ \hdashline 0 & M_2 & 0 & 0 \\ \hdashline 0 & 0 & \ddots & 0 \\ \hdashline 0 & 0 & 0 & M_n \end{array}\right]$$

$$[M_i] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & I_i \end{bmatrix}$$

Calculation of vector generalized forces{Fe}

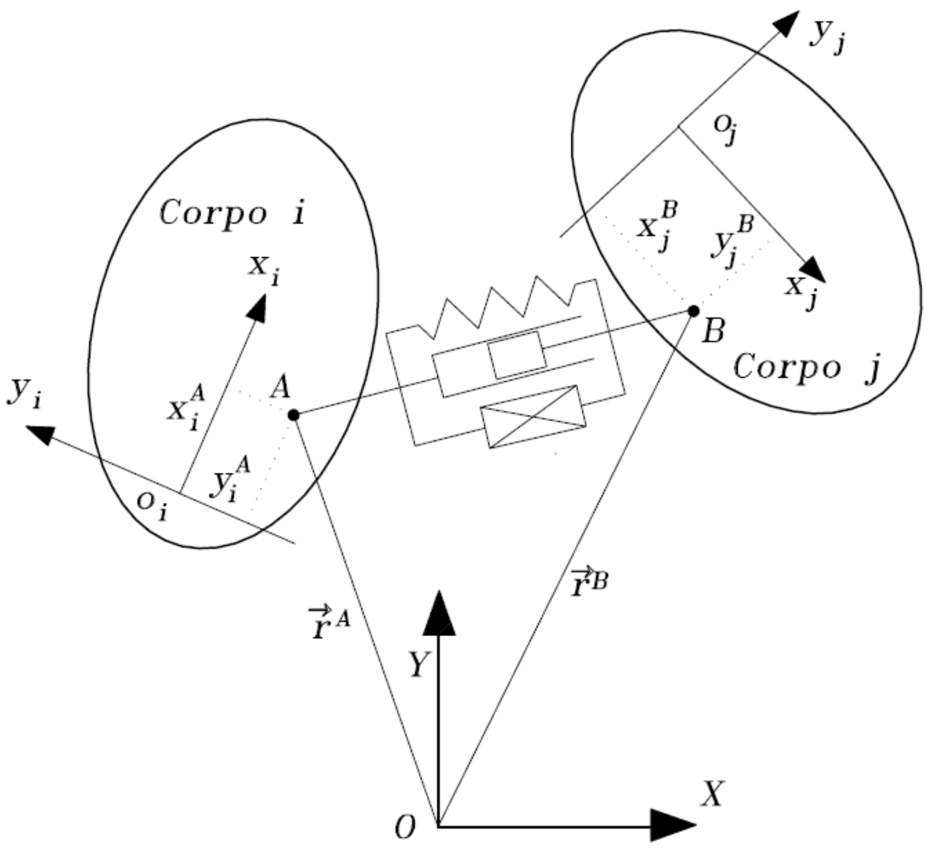
$$\{F_e\} = \left\{ \begin{array}{c} F_x^P \\ F_y^P \\ \{s_i^P\}^T [B_i]^T \{F^P\} \end{array} \right\}$$



$$\{F^P\} = [A_i]\{F_i^P\}$$

$$\{F_e\} = \left\{ \begin{array}{c} \{[A_i]\{F_i^P\}\} \\ \{s_i^P\}^T [B_i]^T [A_i]\{F_i^P\} \end{array} \right\}$$

Spring-damper-element linear actuator



$$\mathcal{F} = k\left(\ell - \ell_0\right) + c\dot{\ell} + \mathcal{F}_a$$

Spring-damper-element linear actuator

- L = AB
- - k : Stiffness coefficient;
- - c : damping coefficient (viscoso);
- - Fa : law of arbitrary force;
- - d_{AB} = AB.

$$\{Q_i\} = \frac{\mathcal{F}}{\ell} \left\{ \begin{array}{c} \{d_{AB}\} \\ \{d_{AB}\}^T [B_i] \left\{ \begin{array}{c} x_i^A \\ y_i^A \end{array} \right\} \end{array} \right\},$$

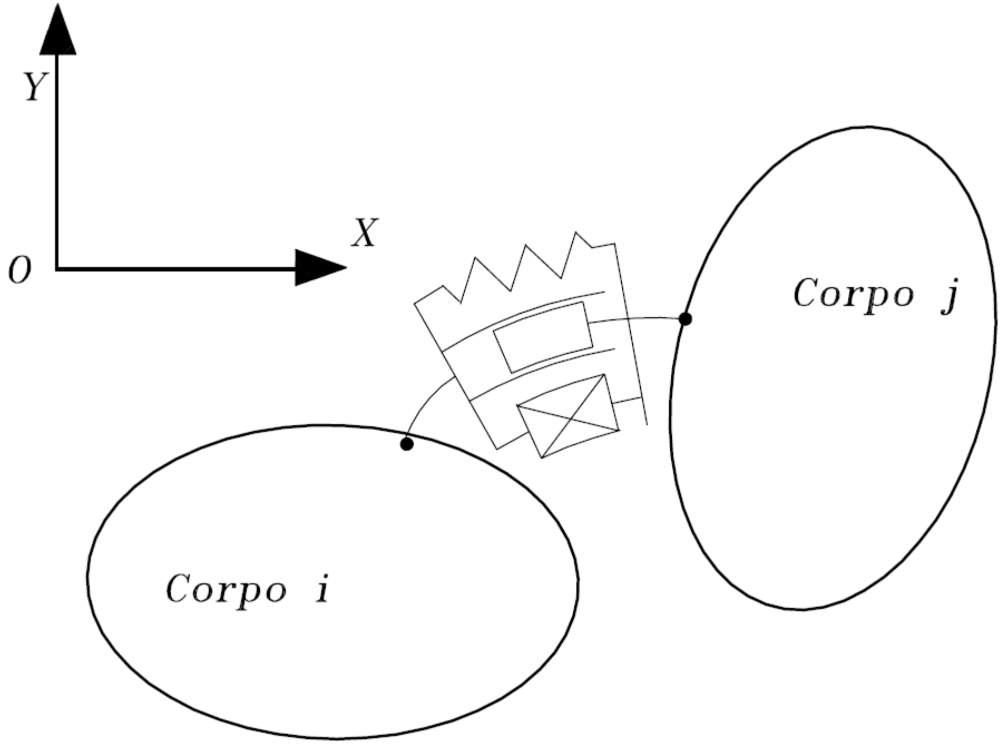
$$\{Q_j\} = -\frac{\mathcal{F}}{\ell} \left\{ \begin{array}{c} \{d_{AB}\} \\ \{d_{AB}\}^T [B_j] \left\{ \begin{array}{c} x_j^B \\ y_j^B \end{array} \right\} \end{array} \right\}$$

$$[B_i] = \left[\begin{array}{cc} -\sin q_{3i} & -\cos q_{3i} \\ \cos q_{3i} & -\sin q_{3i} \end{array} \right].$$

Element spring-damper-actuator angular

$$\vartheta_{ij} = q_{3j} - q_{3i}$$

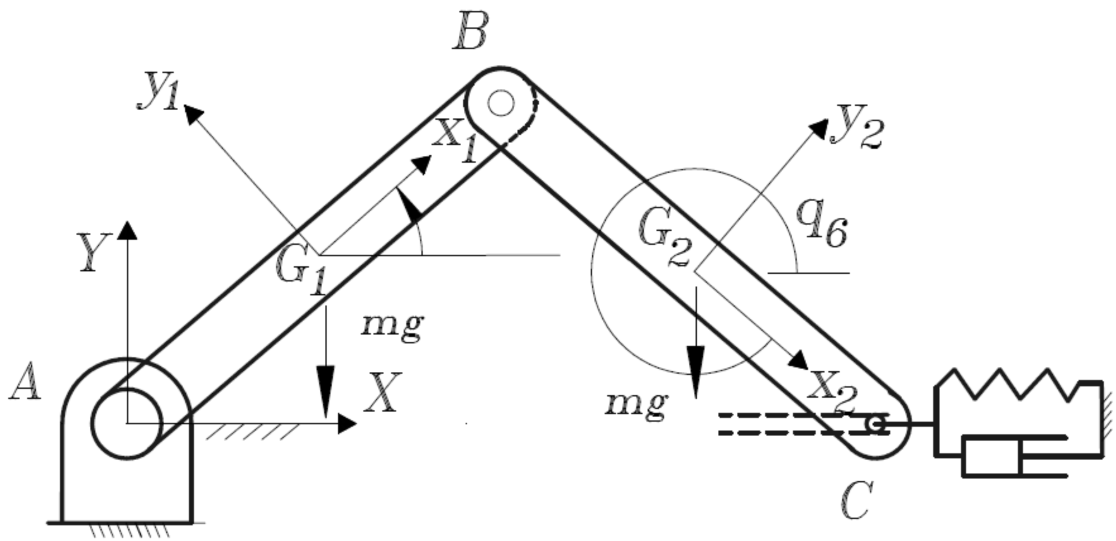
$$\mathcal{T} = k_\vartheta \left(\vartheta_{ij} - \vartheta_{ij}^0\right) + c_\vartheta \dot{\vartheta}_{ij} + \mathcal{T}_a \ ,$$



$$\{Q_i\} = \begin{Bmatrix} 0 \\ 0 \\ \mathcal{T} \end{Bmatrix} \ ,$$

$$\{Q_j\} = -\begin{Bmatrix} 0 \\ 0 \\ \mathcal{T} \end{Bmatrix} \ .$$

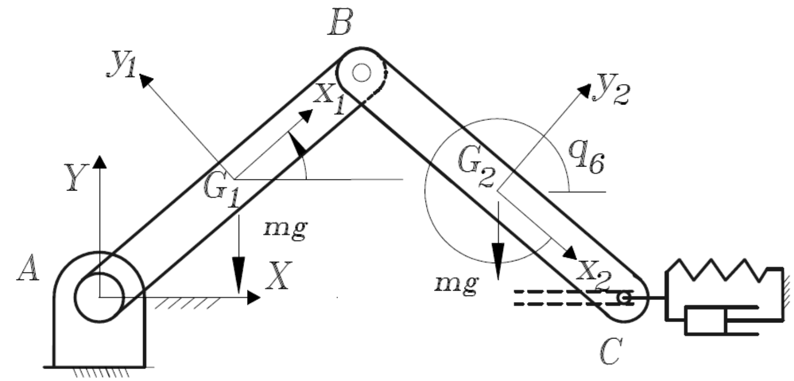
Application



$$X_1^B - X_2^B = 0$$
$$Y_1^B - Y_2^B = 0$$
$$X_1^A - X_3^A = 0$$
$$Y_1^A - Y_3^A = 0$$
$$Y_2^C = 0$$

$$X_1^A = q_1 - L\cos q_3$$
$$Y_1^A = q_2 - L\sin q_3$$
$$X_2^B = q_4 - L\cos q_6$$
$$Y_2^B = q_5 - L\sin q_6$$
$$X_3^A = Y_3^A = 0\ ,$$
$$Y_2^C = q_5 + L\sin q_6$$

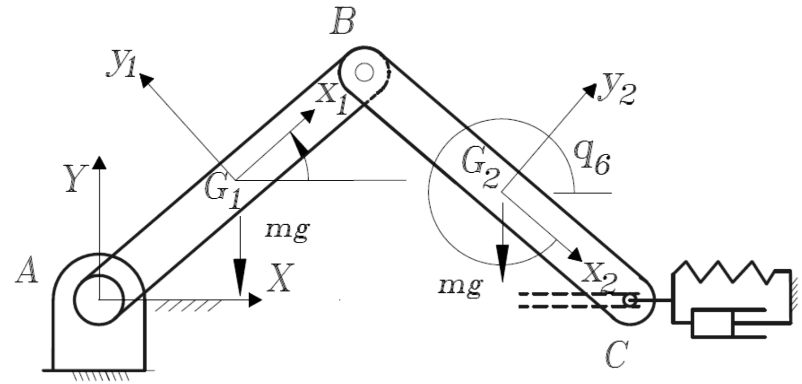
Application



$$\{\Psi\} = \begin{Bmatrix} q_1 - L\cos q_3 \\ q_2 - L\sin q_3 \\ q_4 - L\cos q_6 - q_1 - L\cos q_3 \\ q_5 - L\sin q_6 - q_2 - L\sin q_3 \\ q_5 + L\sin q_6 \end{Bmatrix}$$

$$[\Psi_q] = \begin{bmatrix} 1 & 0 & L\sin q_3 & 0 & 0 & 0 \\ 0 & 1 & -L\cos q_3 & 0 & 0 & 0 \\ -1 & 0 & L\sin q_3 & 1 & 0 & L\sin q_6 \\ 0 & -1 & -L\cos q_3 & 0 & 1 & -L\cos q_6 \\ 0 & 0 & 0 & 0 & 1 & L\cos q_6 \end{bmatrix}$$

Application



$$\{\gamma\} = \left([\Psi_q]\{\dot{q}\}\right)_q\{\dot{q}\} = \left\{\begin{array}{c} -L\cos q_3 \cdot \dot{q}_3^2 \\ -L\sin q_3 \cdot \dot{q}_3^2 \\ -L\cos q_3 \cdot \dot{q}_3^2 - L\cos q_6 \cdot \dot{q}_6^2 \\ -L\sin q_3 \cdot \dot{q}_3^2 - L\sin q_6 \cdot \dot{q}_6^2 \\ L\sin q_6 \cdot \dot{q}_6^2 \end{array}\right\}$$

If q3=3,1415/4 and L=1, then

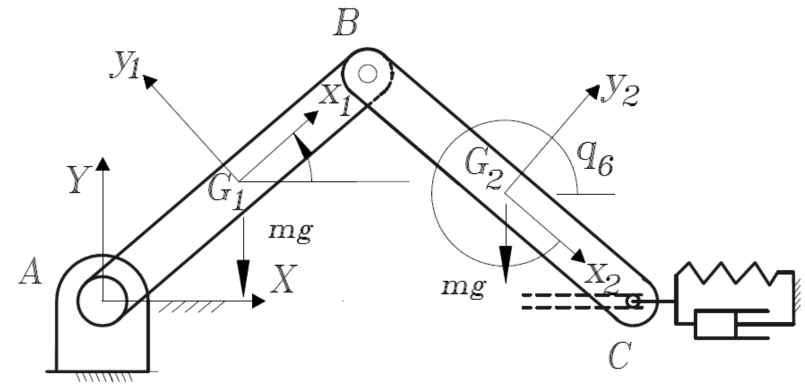
$$q_1 = q_2 = q_5 = \frac{\sqrt{2}}{2}$$

$$q_4 = \frac{3\sqrt{2}}{2},$$

$$q_6 = \frac{7\pi}{4}\text{rad}$$

$$[M] = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

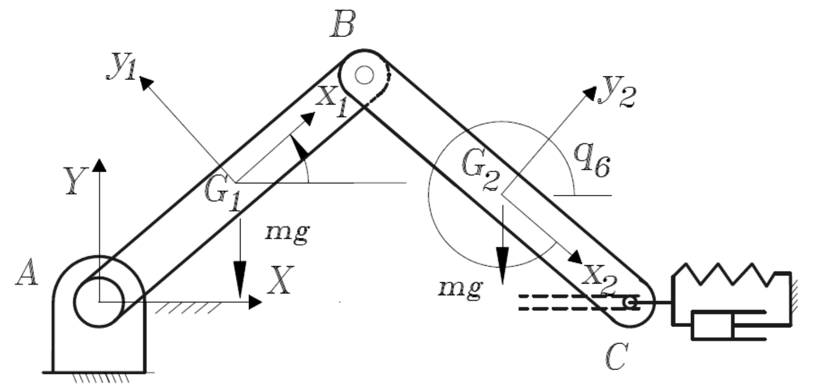
Application



$$\delta W = \{P_1\}^T\{\delta G_1\} + \{P_2\}^T\{\delta G_2\} + \{F_s\}^T\{\delta C\} \ ,$$

$$\{P_1\}^T = \{\ 0 \quad -mg\ \} \ ,$$
$$\{P_2\}^T = \{\ 0 \quad -mg\ \} \ ,$$
$$\{F_s\}^T = \{\ -k\left(X_2^C - \ell_0\right) - c\dot{X}_2^C \quad 0\ \}$$
$$\{\delta G_1\} = \{\ \delta q_1 \quad \delta q_2\ \}^T \ ,$$
$$\{\delta G_2\} = \{\ \delta q_4 \quad \delta q_5\ \}^T \ ,$$
$$\{\delta C\} = \{\ \delta q_4 - L\sin q_6 \delta q_6 \quad 0\ \}^T \ ,$$

Application

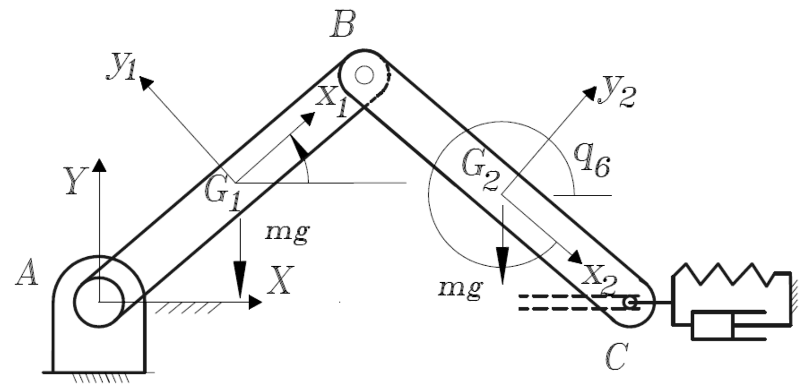


$$\delta W = -mg\left(\delta q_2 + \delta q_5\right) - \left[k\left(q_4 + L\cos q_6 - \ell_0\right) + c\left(\dot{q}_4 - L\sin q_6 \cdot \dot{q}_6\right)\right] \cdot \left(\delta q_4 - L\sin q_6 \delta q_6\right) \; .$$

$$\{F\} = \left\{ \begin{array}{cccc} \frac{\delta W}{\delta q_1} & \frac{\delta W}{\delta q_2} & \cdots & \frac{\delta W}{\delta q_6} \end{array} \right\}^T$$

$$\{F\} = \left\{ \begin{array}{c} 0 \\ -mg \\ 0 \\ -k\left(q_4 + L\cos q_6 - \ell_0\right) - c\left(\dot{q}_4 - L\sin q_6 \cdot \dot{q}_6\right) \\ -mg \\ \left[k\left(q_4 + L\cos q_6 - \ell_0\right) + c\left(\dot{q}_4 - L\sin q_6 \cdot \dot{q}_6\right)\right] L\sin q_6 \end{array} \right\}$$

Application



$$\begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ \gamma \end{Bmatrix}$$

For t=0:

$$\begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix}_0 = \begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix}_0^{-1} \begin{Bmatrix} F \\ \gamma \end{Bmatrix}_0$$